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THE EXISTENCE OF NONMINIMAL SOLUTIONS TO THE YANG-MILLS EQUATION WITH GROUP SU(2) ON $S^2 \times S^2$ AND $S^1 \times S^3$

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Abstract

By generalizing Taubes' approach in [19], we construct an infinite number of gauge inequivalent irreducible SU(2)-connections over $S^2 \times S^2$ and $S^1 \times S^3$, which are nonminimal solutions to the Yang-Mills equations. These connections have a uniform background curvature, with concentrations near points, spaced evenly along a geodesic. Near half of these points the solution looks self-dual, and near the other half it looks anti-self-dual.

1. Introduction

Consider the Yang-Mills equations on a compact, oriented 4-dimensional Riemannian manifold M as the variational equations of a functional YM. The function space $\mathscr{B}$ is the space of isomorphism classes of pairs (P, A), where P is a principal G-bundle, $P \to M$, and A is a smooth connection on P. With respect to the C^∞-topology, $\mathscr{B} = \bigcup_n \mathscr{B}_n$ is the disjoint union of the spaces $\mathscr{B}_n$ which are indexed by $n \in \mathscr{Z}$. The integer n is minus the second Chern number $P \times_{SU(2)} \mathscr{C}^2$. (This is the physicist's instanton number.)

Having fixed the Riemannian metric on the tangent space TM, the Yang-Mills functional is a natural, nonnegative functional on $\mathscr{B}$; this is an energy functional which measures the amount that a given connection's horizontal subbundle in TP fails to be involutive. It assigns to an orbit $[A] \in \mathscr{B}$ of a connection A the number

$$\text{YM}(A) = \tfrac{1}{2} \int_M |F_A|^2 \, dv . \tag{1.1}$$

Here F_A is the curvature of the connection A, a section over M of the vector bundle $\Omega^2(\operatorname{Ad} P) = \operatorname{Ad} P \otimes \bigwedge^2 T^* M$, and $\operatorname{Ad} P$ is the associated vector bundle, $\operatorname{Ad} P = P \times_{\operatorname{Ad}} L(G)$ ($L(G)$ is the Lie algebra of G).

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The norm above is that which is induced from the standard metrics inner product on TM, and a Killing form on $L(G) \equiv \mathrm{Lie\,alg}(G)$.

On $\mathscr{B}_n$, the Yang-Mills functional takes values in $[8\pi^2|n|, \infty)$. The functional may take on its minimal value $8\pi^2|n|$; these minimal points are precisely the set of points in $\mathscr{B}_n$, that are orbits of connections whose curvatures are self- (anti-self) dual with respect to the Hodge star operator on $\bigwedge^2 T^*M$. (The Hodge star operator $*\colon \bigwedge^p T^* \to \bigwedge^{4-p} T^*M$ is uniquely defined by the requirement that for each p-form ω, $\omega \wedge * \omega = (\omega, \omega)\, dv$, where $(\cdot, \cdot)$ is the metric on $\bigwedge^p T^*M$, and dv is the metrics volume 4-form.) An orbit $[A]$ in $\mathscr{B}$ of a connection A lies in the set of minimal points if and only if the curvature of A satisfies

$$F_A = \pm * F_A, \tag{1.2}$$

where $\pm$ depends on $\pm n > 0$. The set is called the moduli space of self-(anti-self) dual connections. For details on the above, we refer to [3], [6], and [9] or [10].

One of the problems in Yang-Mills theory is to find solution to the Yang-Mills equations. Since the Yang-Mills functional does not obey a Palais-Smale condition (the variational equations are an semilinear partial differential equations with critical exponent like the Yamabe equation), we cannot directly use the Ljusternik-Snirelman argument. The failure of the Palais-Smale condition is not always the final word. C. H. Taubes in [17]–[19] constructed many minimal solutions (self-dual or anti-self-dual connections) to the Yang-Mills equations on general, compact, oriented 4-manifolds by using the method of small eigenvalues. The lesson to be learned from Taubes' construction about minimal solutions is the following one: One may find solutions without the Palais-Smale condition.

The purpose of this paper is to find nonminimal solutions to the Yang-Mills equations. Thus, we find a connection A on a principal G-bundle P, whose curvature F_A satisfies

$$D_A^* F_A = 0. \tag{1.3}$$

Here D_A^* is the formal L^2-adjoint of D_A, and D_A is the covariant exterior derivative defined by A. The connections that we find are neither self-dual nor anti-self-dual. For $M = S^2 \times S^2$, $M = S^1 \times S^3$, and the SU(2) structure group G, we obtain the following theorems.

Theorem 1.1. *Let (m, n) be a pair of integers which obeys the following conditions:*

(1) $|m| \neq |n|$.

(2) *If* $|m| > |n|$, *then* $|m| \neq |n|(2l+1) + l(l+1)$ *for* $l = 0, 1, 2, \cdots$.

(3) *If* $|n| > |m|$, *then* $|n| \neq |m|(2l+1)+l(l+1)$ *for* $l = 0, 1, 2, \cdots$.
Then there exists a positive integer $K_0 > 0$ *such that for any positive odd number* $k > K_0$, *there exists an irreducible* SU(2)*-connection* $A(m, n, k)$ *over* $S^2 \times S^2$ *with degree* $2mn$ *which is a nonminimal solution to the Yang-Mills equations, and its action obeys*

$$\mathrm{YM}(A(m, n, k)) \in (8\pi^2(m^2+n^2+2k)-\varepsilon, 8\pi^2(m^2+n^2+2k)+\varepsilon)$$

for some $\varepsilon \leq 1$.

Theorem 1.2. *If the base manifold is* $S^1 \times S^3$, *then there is a positive integer* $K_0 > 0$ *such that for any positive odd number* $k > K_0$, *there exists an irreducible* SU(2)*-connection* $A(k)$ *over* $S^1 \times S^3$ *with degree zero which is a nonminimal solution of the Yang-Mills equations, and its action is* $\mathrm{YM}(A(k)) > 16\pi^2 k$.

Remark 1.1.. None of the solutions found above are symmetric with respect to the Lie groups actions on $S^2 \times S^2$ or $S^1 \times S^3$. In fact, our solutions have the property that there is a set of points about which curvature concentrates.

Remark 1.2. We conjecture that the theorems above have analogs for other 4-manifolds which can be proved using the techniques which we introduce.

Recently, L. Sibner, R. Sibner, and K. Uhlenbeck have found an infinite number of nonminimal solutions to the Yang-Mills equations over S^4 by a min-max argument (cf. [16]). Parker [15] has studied symmetric solutions on homogeneous 4-manifolds.

The strategy for proving the theorems generalizes the approach in [19]–[21]. Schematically, the approach is as follow: The assignment of $[A] \in \mathscr{B}$ to $D_A^* F_A$ defines a vectorfield $\nabla \mathrm{YM}$, the tangent bundle $T\mathscr{B} \to \mathscr{B}$. The problem is to determine when this vectorfield has a zero at which the Yang-Mills functional takes a nonminimal value.

The cut and paste operation in [19]–[21] constructs a finite-dimensional manifold N with an embedding $i: N \to \mathscr{B}$. The manifold N has the property that the norm of $\nabla \mathrm{YM}$ is small (N is called the end point set of the Yang-Mills functional, and will be described shortly).

The eigenvectors with small eigenvalues of the Hessian $\nabla^2 \mathrm{YM}$ (the second variation) of the Yang-Mills functional are obstructions to a direct application of the implicit function theorem to perturb N into the zeros of $\nabla \mathrm{YM}$. However, at each $y \in N$, the Hessian of the functional YM has only a finite-dimensional eigenvector subspace with small eigenvalues, and all other eigenvalues are $\mathscr{O}(1)$. The eigenvector subspace with small eigenvalues at each $y \in N$ defines a vector bundle $V \to N$ as a subbundle of $i^* T\mathscr{B}$.

As Taubes did in [19], we use a global version of Kuranishi's ideas (on complex structure deformations) to construct a section $f: N \to V$ such that the zero points of f are contained in the zero point set of ∇ YM on which the Yang-Mills functional takes nonminimal values. The remainder of this article is devoted to the construction of N and to finding nonminimal solutions to the Yang-Mills equations on $S^2 \times S^2$ and $S^1 \times S^3$.

Our construction of the approximate solution space was inspired by Wente's [25] solution of the Hopf conjecture.

2. Basic notions

Let G be a compact, simple Lie group, and $P \to M$ a principal G-bundle, where M is a compact, oriented 4-manifold. Let $\mathscr{C}(P)$ denote the space of all smooth connections on P. Fix $A_0 \in \mathscr{C}(P)$. As $\mathscr{C}(P)$ is an affine space, any connection $A \in \mathscr{C}(P)$ can be written uniquely as $A = A_0 + a$ with $a \in \Gamma(\mathrm{Ad}\, P \otimes T^* M)$. The connection A is a Yang-Mills connection if the 1-form a satisfies

$$D^*_{A_0+a} F_{A_0+a} = 0; \tag{2.1}$$

that is,

(2.2)

$$D^*_{A_0} D_{A_0} a + D^*_{A_0}(a \wedge a) - *[a, *D_{A_0} a] - *[a, *(F_{A_0} + a \wedge a)] + D^*_{A_0} F_{A_0} = 0.$$

The problem now is to find $A_0 \in \mathscr{C}(P)$ such that the nonlinear partial differential equation (2.2) has a solution $a \in \Gamma(\mathrm{Ad}\, P \otimes T^* M)$. We call a self-dual (anti-self-dual) solution to (2.1) an instanton (anti-instanton).

The Yang-Mills equations are the Euler-Lagrange equations for the Yang-Mills functional YM (i.e., the variation equations) on the domain $\mathscr{C}(P)$. Now, this domain is contractible, but it is invariant under the gauge group $\mathrm{Aut}\, P$, so one should consider in the induced functional on the quotient space $\mathscr{B}$ $(= \mathscr{C}/\mathrm{Aut}\, P)$. The group $\mathrm{Aut}\, P$ is infinite dimensional, and may be identified with $\Gamma(\mathrm{Ad}\, P)$. Although $\mathrm{Aut}\, P$ does not act freely, its normal subgroup $\mathrm{Aut}\, P(x)$ consisting of based automorphisms (i.e., automorphisms which are the identity over a given point $x \in M$) does, and one has $\mathrm{Aut}\, P/\mathrm{Aut}\, P(x) = G$. For this reason, we consider also the Yang-Mills functional YM on the quotient $\mathscr{B}' = \mathscr{C}/\mathrm{Aut}\, P(x)$.

We will consider the space of L^2_2-connections on P, and denote it also by $\mathscr{C}(P)$ (cf. [10]). The gauge group $\mathrm{Aut}\, P$ will denote the Banach Lie group of L^2_3-automorphisms of P. Then $\mathscr{B}' = \mathscr{C}(P)/\mathrm{Aut}\, P(x)$

is a smooth Banach manifold with an L_2^2-Sobolev space for its model, and $\mathscr{B}$ is defined as a topological space with the quotient topology. The quotient space $\mathscr{B}$ is not quite a Banach manifold, but denote by $R(P)$ the infinite codimensional set of reducible connections on P, and let $\mathscr{C}^\sharp(P) = \mathscr{C}(P)\backslash R(P)$. Then $\mathscr{B}^\sharp = \mathscr{C}^\sharp/\operatorname{Aut} P$ is a smooth Banach manifold, and the map $\mathscr{C}^\sharp \to \mathscr{B}^\sharp$ defines a smooth principal $\operatorname{Aut} P/(\text{center}\, G)$-bundle over $\mathscr{B}^\sharp$.

For $q \in \{0, \cdots, 4\}$, let $\Omega^q(\operatorname{Ad} P)$ denote the vector bundle $\operatorname{Ad} P \otimes \bigwedge^q T^*M$. Fix a smooth connection on P. With $A = A_0$, one defines the L_k^2-Sobolev norm on $\Omega^q(\operatorname{Ad} P)$ as follows. For a section ψ, set

$$\|\psi\|_{L_k^2}^2 = \int_M \sum_{i=0}^{k} |\nabla_A^k \psi|^2, \tag{2.3}$$

where ∇_A is the covariant derivative from the connection A on P and from the given Riemannian metrics Levi-Civita connection on the tensor bundle. Let $L_k^2(\Omega^q(\operatorname{Ad} P))$ denote the Banach space which is obtained by completing the space of the smooth section of $\Omega^q(\operatorname{Ad} P)$ in the norm of (2.3).

With its L_2^2-Sobolev structure, the tangent space to a connection A in $\mathscr{C}(P)$ is precisely $L_2^2(\Omega^q(\operatorname{Ad} P))$. With its L_3^2-Sobolev structure, the Lie algebra of the Banach Lie group $\operatorname{Aut} P$ is $L_3^2(\operatorname{Ad} P)$.

The tangent space to $\mathscr{B}^\sharp$ is the Banach manifold

$$T\mathscr{B}^\sharp = \{(A, a) \mid A \in \mathscr{C}^\sharp(P), \text{ and } a \in L_2^2(\Omega^1(\operatorname{Ad} P)) \text{ satisfies } D_A^* a = 0\}/\operatorname{Aut} P. \tag{2.4}$$

For a connection A in $\mathscr{C}(P)$, its curvature F_A is in $L_1^2(\Omega^1(\operatorname{Ad} P))$. Thus, the Yang-Mills functional in (1.1) is finite on $\mathscr{B}$, and one can check easily that it is smooth on $\mathscr{B}^\sharp$. It is convenient to consider the infinite-dimensional vector bundle over $\mathscr{B}'$, $V' \to \mathscr{B}'$, which is defined to be the vector bundle $(\mathscr{C}(P)) \times L_2^2(\Omega^1(\operatorname{Ad} P))/\operatorname{Aut} P(x)$. There is a natural G-action on V', which factors through $G/\operatorname{center}(G)$ and covers the action on $\mathscr{B}$. Let $V = V'/G$. Over $\mathscr{B}^\sharp$d, V is a smooth vector bundle. The tangent bundle of $\mathscr{B}^\sharp$ now appears as a closed subbundle of V. Likewise, the tangent bundle of $\mathscr{B}'$ is a closed, G-invariant subbundle of V'. The vector bundle V' has a convenient, G-invariant fiber metric: Let $u = [A, a]$ and $v = [A, b]$ be two points in V' over A in $\mathscr{B}'$. Then set

$$\langle u, v\rangle_{[A]} = \int_M \{(\nabla_A a, \nabla_A b) + (a, b)\}. \tag{2.5}$$

The Hilbert space $L_{1A} = \{u \in L_1^2(\Omega^1(\operatorname{Ad} P)) \mid D_A^* u = 0\}$ with the inner product $\langle u, v\rangle_{[A]}$ by (2.5) is a closed subspace of $L_1^2(\Omega^1(\operatorname{Ad} P))$. It should be noted that for a gauge transformation $g \in \operatorname{Aut} P$, $L_{1g\cdot A} = g \cdot L_{1A}$.

The affine structure of the space of connections induces a smooth map $f\colon V' \to \mathscr{B}'$ which is the canonical projection when restricted to the canonical zero section of V'. This map sends $v = [A, a]$ to $f(v) = [A + a]$.

Using the map f, the first variation of the functional YM defines a smooth section $\nabla \mathrm{YM}$ of V'^* in the following way: Let $v = [A, a]$ be a point in V' over $\mathscr{B}$. Then

$$\nabla \mathrm{YM}_A(v) = \frac{d}{dt} \mathrm{YM}(A + ta)|_{t=0} = \int_M (F_A, D_A a). \tag{2.6}$$

A point $[A]$ in $\mathscr{B}'$ is a critical point when $\nabla \mathrm{YM}_A(\cdot) = 0$. The norm of (2.5) induces a G-invariant norm on the dual space V'^*, and it is this dual norm which will be used to measure the size of $\nabla \mathrm{YM}$ at the point in $\mathscr{B}'$. Since $\nabla \mathrm{YM}$ is G-equivariant, $\nabla \mathrm{YM}$ descends to define $\nabla \mathrm{YM}\colon \mathscr{B} \to V^*$, with the assignment of $\|\nabla \mathrm{YM}\|_{[A]}^*$ to $[A] \in \mathscr{B}$ defining a continuous function.

The Hessian $\nabla^2 \mathrm{YM}$ of YM which is nominally only well defined at the critical points of YM, can be extended, using the map f, to a smooth section over $\mathscr{B}'$ of the vector bundle $\operatorname{Sym}^2 V'^*$. Let $v = [A, a]$ be a point in V' over $[A]$ in $\mathscr{B}'$. Then

$$\nabla^2 \mathrm{YM}_A(v) = \frac{d^2}{dt^2} \mathrm{YM}(A + ta)|_{t=0}. \tag{2.7}$$

The size of $\nabla^2 \mathrm{YM}$ at each point in $\mathscr{B}'$ will be measured using the norm on $\operatorname{Sym}^2 V'^*$ which is induced from the norm in (2.5) on V'. It is easily verifiable that the Hessian at $[A]$ of YM on the fiber of V' viewed as a symmetric bilinear form is

$$\nabla^2 \mathrm{YM}_A(a, b) = \int_M \{(D_A a, D_A b) + (F_A, a \wedge b + b \wedge a)\}, \tag{2.8}$$

where $[A, a]$ and $[A, b] \in V'$.

As $*^2 = 1$ on $\bigwedge^2 T^*M$, $*$ induces the decomposition

$$\textstyle\bigwedge^2 T^*M = P_+ \bigwedge^2 T^*M \oplus P_- \bigwedge^2 T^*M,$$

where $P_\pm = \frac{1}{2}(1 \pm *)$. Write $F_\pm = P_\pm F_A$ and define operators

$$P_\pm D_A\colon \Gamma(\operatorname{Ad} P \otimes T^*M) \to \Gamma(\operatorname{Ad} P \otimes P_\pm \textstyle\bigwedge^2 T^*M).$$

Then the Hessian as defined above can be written as

$$(2.9)\quad \nabla^2 \mathrm{YM}_A(a, b) = 2\int_M \{(P_\pm D_A a, P_\pm D_A b) + (F_\pm, a\wedge b + b\wedge a)\}.$$

In order to relate the topology of $\mathscr{B}$ to the critical points of YM, the points where $\nabla \mathrm{YM} = 0$, one needs some conditions on YM, which require the gradient (the first order Taylor's expansion) to offer an approximation to the functional on a uniform neighborhood of any given point. Using the norm in (2.5), the following propositions describe the differentiability of the functional YM.

Proposition 2.1. *As a map* $\mathscr{C}(P) \times L_1^2(\mathrm{Ad}\, P \otimes T^*M)$ *to* $[0, \infty)$, *the assignment* $(A, u) \to \|u\|_A^2$ *is smooth. In addition, there exists* $C < \infty$ *which is independent of* (A, u) *such that* $\|u\|_{L^4}^2 \le C\|u\|_A^2$. *Further, for all* $(A, u, a) \in \mathscr{C}(P) \times_2 L_1^2(\mathrm{Ad}\, P \otimes T^*M)$,

$$(2.10)\qquad |\,\|u\|_{A+a} - \|u\|_A| \le 4\|u\|_{L^4}\|a\|_A \le 4C^2\|u\|_A\|a\|_A.$$

Proposition 2.2. *The Yang-Mills functional* YM *is smooth on the affine space* $\mathscr{C}(P)$ *(with* L_1^2*-Sobolev structure), and there is a constant* $C < \infty$ *which is independent of* $A \in \mathscr{C}(P)$ *and* $a, u, v \in L_1^2(\mathrm{Ad}\, P \otimes T^*M)$ *such that*

$$(2.11)\quad \begin{aligned} &(1)\quad |\mathrm{YM}(A+a) - \mathrm{YM}(A)| \le C\|a\|_A(1 + \|a\|_A^3),\\ &(2)\quad |\mathrm{YM}(A+a) - \mathrm{YM}(A) - \nabla \mathrm{YM}_A(a)| \le C\|a\|_A^2(1 + \|a\|_A^2),\\ &(3)\quad |\mathrm{YM}(A+a) - \mathrm{YM}(A) - \nabla \mathrm{YM}_A(a) - \tfrac{1}{2}\nabla^2 \mathrm{YM}_A(a, a)|\\ &\qquad\qquad \le C\|a\|_A^3(1 + \|a\|_A), \end{aligned}$$

$$(2.12)\quad \begin{aligned} &(1)\quad |\nabla \mathrm{YM}_{A+a}(u) - \nabla \mathrm{YM}_A(u)|\\ &\qquad\qquad \le C(1 + \mathrm{YM}(A))^{1/2}\|u\|_A\|a\|_A(1 + \|a\|_A^2),\\ &(2)\quad |\nabla^2 \mathrm{YM}_{A+a}(u, v) - \nabla^2 \mathrm{YM}_A(u, v)|\\ &\qquad\qquad \le C(1 + \mathrm{YM}(A))^{1/2}\|u\|_A\|v\|_A\|a\|_A(1 + \|a\|_A),\\ &(3)\quad |\nabla \mathrm{YM}_{A+a}(u) - \nabla \mathrm{YM}_A(u) - \nabla^2 \mathrm{YM}_A(a, u)|\\ &\qquad\qquad \le C(\|a\|_A^2 + \|a\|_A^3)\|u\|_A. \end{aligned}$$

The proofs are in [20] and [21].

(2.2) can be written in the following way. Let A be a smooth connection on P (in general, require that $\nabla \mathrm{YM}_A$ is of small dual norm). Corresponding to (2.2), we have

$$(2.13)\qquad \nabla \mathrm{YM}_{A+a}(\cdot) = 0;$$

that is,

$$\nabla \mathrm{YM}(\cdot) + \nabla^2 \mathrm{YM}_A(a, \cdot) + R(a, \cdot) = 0. \tag{2.14}$$

In (2.14), $R(a, \cdot)$ is the remainder coming from the second-order Taylor's expansion of $\nabla \mathrm{YM}$, and satisfies the estimate $\|R(a, \cdot)\|_A^* \le C(\|a\|_A^2 + \|a\|_A^3)$; this uses (2.12). If $a \in \Gamma(\mathrm{Ad}\, P \otimes T^*M)$ satisfies (2.14), then $A + a$ is a Yang-Mills connection on P.

Now we turn our attention to the Hessian $\nabla^2 \mathrm{YM}$ of YM. Proposition 2.2 shows that $\nabla^2 \mathrm{YM}$ defines a bounded, symmetric, bilinear form on the fiber of V over $[A]$, and the standard elliptic theory implies that it is a closed form.

A real number ρ is said to be in the resolvent set of $\nabla^2 \mathrm{YM}$ if the quadratic form $\nabla^2 \mathrm{YM} - \rho\langle\cdot, \cdot\rangle_A$ is nondegenerate. Any number which is not in the resolvent set of $\nabla^2 \mathrm{YM}$ is said to be in the spectrum of $\nu^2 \mathrm{YM}$. An eigenvector of $\nabla^2 \mathrm{YM}$ with eigenvalue ξ is a nonzero vector in the fiber of V over $[A]$ with the property that $\nabla^2 \mathrm{YM}_A(v, \cdot) - \xi\langle v, \cdot\rangle_A = 0$.

The Hessian $\nabla^2 \mathrm{YM}$ defines a Fredholm operator only if its domain is restricted. This has to be done because the functional is gauge invariant. On our space V', $\nabla^2 \mathrm{YM}$ has an infinite null space due to vectors tangent to the gauge orbit. To write down the restricted domain, we require some ideas from [21, §§6–8].

To begin, for each $\nu \ge 0$ and for each L_1^2-connection A on P, introduce $\mathscr{G}_\nu$, the linear span of the L^2-eigenvectors of $\nabla_A^* \nabla_A$ on $L^2(\mathrm{Ad}\, P)$ with eigenvalues in the interval $[0, \nu]$ (it is possible that $\mathscr{G}_\nu = 0$). In particular, when $[A] \in \mathscr{B}^\sharp$, $\mathscr{G}_0[A] = 0$.

Define $\mu_\nu[A]$ to be the difference between the first eigenvalue of the unbounded operator $\nabla_A^* \nabla_A$ on $L^2(\mathrm{Ad}\, P)$ in the interval (ν, ∞) and ν. For $\nu = 0$, this is the first nonzero eigenvalue of $\nabla_A^* \nabla_A$.

Let A be an L_1^2-connection on P, and define

$$L_{\nu 1 A} = \{v \in L_1^2(\Omega^1(\mathrm{Ad}\, P)) \mid D_A^* v \in \mathscr{G}_\nu[A]\}. \tag{2.15}$$

This is a Hilbert space with the inner product of (2.15). When $g \in \mathrm{Aut}\, P$, $\mathscr{G}_\nu[g \cdot A] = g \cdot \mathscr{G}_\nu[A]$ and so $l_{\nu 1 g \cdot A} = g \cdot L_{\nu 1 A}$.

Restrict $\nabla^2 \mathrm{YM}$ to $L_{\nu 1 A}$ and define a bilinear form. This bilinear form is closed, and its spectrum on $L_{\nu 1 A}$ in $(-\infty, 1)$ is pure point spectrum. Furthermore, its eigenvalues have finite multiplicities, and the only accumulation point in $[-\infty, 1]$ is the number 1 (for details on these, see [21, §7]).

Now, let $\xi \in (-\infty, 1)$. Let $\pi_\nu(A, \xi)\colon L_{\nu 1A} \to L_{\nu 1A}$ be the orthogonal projection onto the space spanned by eigenvectors of $\nabla^2 \mathrm{YM}$ on $L_{\nu 1A}$ with eigenvalue $< \xi$. When ξ_0 is not the spectrum of $\nabla^2 \mathrm{YM}_{A_0}$ on $L_{\nu_0 q A_0}$, $\pi_\nu(A, \xi)$ is continuous as ξ varies near ξ_0, as ν varies near ν_0, and as $[A]$ varies near $[A_0]$ in $\mathscr{B}'$ (for details, see [21, §§6–7]).

Divide (2.13) into two parts: First take $\xi < 1$ to be a small positive real number such that $\pm\xi$ are not in the spectrum of $\nabla^2 \mathrm{YM}$. Then consider

$$(2.16)\qquad (1 - \pi_\nu(A + a, -\xi))^* \cdot \pi_\nu(A + a, \xi)^* \nabla \mathrm{YM}_{A+a}(\cdot) = 0,$$

$$(2.17)\qquad \{(1 - \pi_\nu(A + a, \xi))^* + \pi_\nu(A + a, -\xi)^*\} \nabla \mathrm{YM}_{A+a}(\cdot) = 0,$$

where 1 denotes the identity. If $a \in \Gamma(\mathrm{Ad}\, P \otimes T^*M)$ satisfies (2.16) and (2.17) simultaneously, then $A + a$ is a Yang-Mills connection.

To solve (2.17), we make use of a map $\psi_\nu[A, a]\colon L_{\nu 1A} \to L_{\nu 1A+a}$ with the following properties.

Lemma 2.1. *Fix a principal G-bundle $P \to M$. Let A be a connection on P. Let $\nu \geq 0$ and $\mu_\nu[A] > 0$ be given. Then there exist $\varepsilon(\mathrm{YM}(A), \nu, \mu_\nu[A]) > 0$ and $Z \equiv Z(\mathrm{YM}(A), \nu, \mu_\nu[A]) < \infty$ with the following significance: Let $a \in L_{\nu 1A}$ obey $\|a\|_A \leq \varepsilon$. Then there exists*

$$\psi_\nu[A, a]\colon L_{\nu 1A} \to L_{\nu 1A+a}$$

which is 1-1, *onto, and is such that for each $v \in L_{\nu 1A}$ the following estimates are satisfied:*

(1) $\big| \|\psi_\nu[A, a] \cdot v\|_{A+a} - \|v\|_A \big| \leq Z \|v\|_A \cdot \|a\|_A$.
(2) $\|\psi_\nu[A, a] \cdot v - v\|_A \leq Z \|v\|_A \cdot \|a\|_A$.
(3) *For each* $g \in \mathrm{Aut}\, P$, $\psi_\nu[g \cdot A, g \cdot a] \cdot (g \cdot v) = g \cdot \psi_\nu[A, a] \cdot v$.

For details on the above lemma, see §6 of [21].

(2.17) for a is equivalent to the vanishing on $L_{\nu 1A}$ of the linear functional

$$(2.18)\qquad \begin{aligned} &\nabla \mathrm{YM}_{A+a}(\{(1 - \pi_\nu(A + a, \xi)) + \pi_\nu(A + a, -\xi)\} \\ &\qquad \circ \psi_\nu[A, a] \circ \{(1 - \pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot)), \end{aligned}$$

which is equivalent to

$$(2.19)\qquad \begin{aligned} &\nabla \mathrm{YM}_A(\{(1 - \pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot)) \\ &\quad + \nabla^2 \mathrm{YM}_A(a, \{(1 - \pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot)) \\ &\quad + R_\nu(A, \xi; a)(\{(1 - \pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot)) \\ &= 0. \end{aligned}$$

When a is sufficiently small, $R(A, \xi; a)(\{(1-\pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot))$ has the following estimate: For any $v \in L_{\nu 1A}$

$$\begin{aligned}(2.20)\qquad &|R_\nu(A, \xi; a)(\{(1-\pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(v))| \\ &\quad \le Z \cdot (\|a\|_A + \|\nabla \mathrm{YM}_A(\cdot)\|_A^*) \cdot \|a\|_A \cdot \|v\|_A,\end{aligned}$$

where Z depends only on ξ, μ_ν, and $E = \mathrm{YM}(A)$.

Proposition 2.3. *Suppose $P \to M$ is a principal G-bundle over a compact, oriented 4-manifold M. Let A be a smooth connection on P at which $\nabla \mathrm{YM}$ is of small norm. Let $\xi < 1$ be a positive real number and suppose $\pm\xi$ are not in the spectrum of $\nabla^2 \mathrm{YM}_A$ on $L_{\nu 1A}$. Then there exists a positive constant C which depends only on ν, $\mu_\nu[A]$, E, and $\Delta_{\nu\xi}[A]$ where $\Delta_{\nu\xi}[A] = \mathrm{dist}_R(\xi, \mathrm{Spec}\, \nabla^2 \mathrm{YM})$, such that if $\|\nabla \mathrm{YM}_A(\cdot)\|_A^* < C$, then there exists a solution a to* (2.17) *which obeys the a priori estimate*

$$(2.21)\qquad \|a\|_A \le Z \|\nabla \mathrm{YM}_A(\{1-\pi_\nu(A, \xi)) + \pi_\nu(A, -\xi)\}(\cdot))\|_A^*,$$

where Z depends only on ν, $\mu_\nu[A]$, E, and $\Delta_{\nu\xi}[A]$. Furthermore, $a(A)$ is equivariant under the action of Aut P; *that is, for $g \in \mathrm{Aut}\, P$, $a(g \cdot A) = g \cdot a(A)$.*

The proof of the above proposition is omitted (refer to [21, §§7–8]).

Remark 2.1. Existence and uniqueness of $a(A)$ follow from the contraction mapping principle by using (2.20). Furthermore, elliptic regularity estimates imply that $a \in L_{\nu 1A}$ which proves (2.17) must be C^∞ when A is a smooth connection. This is proved in §9 of [21].

Proposition 2.3 establishes a map $\mathscr{U}_\xi$ (for $\pm\xi \ne$ eigenvalues of $\nabla^2 \mathrm{YM}_A$ on $L_{\nu 1A}$) which maps

$$\mathscr{C}_\xi = \{A \in \mathscr{C}(P) \mid \|\nabla \mathrm{YM}_A(\cdot)\|_A^* < C(\nu, \mu_\nu[A], \Delta_{\nu\xi}[A], E)\}$$

to

$$L_{\nu 1A} \bigcap \Gamma(\mathrm{Ad}\, P \otimes T^*M).$$

It is Aut P-equivariant; $\mathscr{U}_\xi(g \cdot A) = g \cdot \mathscr{U}_\xi(A)$.

With $\mathscr{U}_\xi$, we can consider (2.16) as a mapping which sends each $A \in \mathscr{C}_\xi$ to the point

$$\begin{aligned}(2.22)\qquad f_\xi(A) &= (1-\pi_\nu(A + \mathscr{U}_\xi(A), -\xi))^* \\ &\quad \circ \pi_\nu(A + \mathscr{U}_\xi(A), \xi)^* \nabla \mathrm{YM}_{A+\mathscr{U}_\xi(A)}(\cdot).\end{aligned}$$

Expanding $\nabla \mathrm{YM}_{A+\mathscr{U}_\xi(A)}(\cdot)$ in $\mathscr{U}_\xi(A)$, one finds

$$\begin{aligned}(2.23)\qquad f_\xi(A) &= \nabla \mathrm{YM}_A((1-\pi_\nu(A, -\xi)) \circ \pi_\nu(A, \xi)(\cdot)) \\ &\quad + R_\nu(A, \xi, \mathscr{U}_\xi(A); (1-\pi_\nu(A, -\xi)) \circ \pi_\nu(A, \xi))(\cdot).\end{aligned}$$

If $A \in \mathscr{C}_\xi$ is a zero point of f_ξ, then $A + \mathscr{U}_\xi(A)$ is a solution to the Yang-Mills equations.

The remainder of this article is composed of three sections. In §3, a family of approximate solutions will be constructed by grafting standard instantons and anti-instantons (of small scale size) over S^4 onto a reducible or irreducible nonminimal background solution to the Yang-Mills equations. The gluing parameters form a finite-dimensional manifold. In §4, we will analyze how small eigenvalues of the Hessian of YM obstruct a deformation of these approximate solutions to true solutions. In this section, we establish that our parameters account for all small eigenvalues of the Hessian. In §5, the proofs of Theorems 1.1 and 1.2 are completed by an argument which shows that the restriction of $f_\xi(\cdot)$ to our parameter space must have a zero. There are three technical appendices.

3. The approximate solutions

In this section and the remaining ones, we consider, for the most part, only the principal SU(2)-bundle over $S^2 \times S^2$ or $S^1 \times S^3$, where S^1, S^2, and S^3 are given their standard metrics with radius 1.

The purpose of this section is to construct a space of approximate solutions N by a gluing operation. Each $A \in N$ will have the norm of $\nabla \mathrm{YM}(\cdot)$ small enough to invoke the results in §2. (The gluing operation of connections onto other connections is also described in [19]–[21] and [9].)

In our special case, the construction takes several standard self-dual and anti-self-dual SU(2)-connections over S^4 whose curvatures are concentrated about the north pole in S^4, and by a cut and paste operation these connections are grafted onto a fixed background connection. The background must be a nonminimal solution to the Yang-Mills equations on $S^2 \times S^2$ or $S^1 \times S^3$. (In Appendix A, we describe a double indexing family of reducible isolated connections over $S^2 \times S^2$ and an irreducible isolated connection over $S^1 \times S^3$ [24].) The grafting occurs at points along a closed geodesic.

Before beginning the graft, we digress to describe the basic instantons and anti-instantons. For this identify $\mathscr{R}^4 = \mathscr{H} =$ quaternions, $\mathrm{SU}(2) =$ unit quaternions, and $L(\mathrm{SU}(2)) = \mathrm{Im}\,\mathscr{H}$. On $\mathscr{R}^4$, define

$$U_1 = \{x \in \mathscr{R}^4 \mid |x| < 1\}, \qquad U_2 = \mathscr{R}^4 \backslash \{0\}. \tag{3.1}$$

Think of $\mathscr{R}^4$ as $S^4 \backslash \{\text{south pole}\}$. Then a principal SU(2)-bundle over

$\mathscr{R}^4$, $\widehat{P}_+ \to \mathscr{R}^4$, is defined by giving the transition function

$$g_{12}\colon U_1 \bigcap U_2 \to \mathrm{SU}(2), \qquad g_{12}(x) = x/|x|. \tag{3.2}$$

A connection $W \in \mathscr{C}(\widehat{P}_+)$ is specified by data consisting of a pair of $L(\mathrm{SU}(2))$-valued 1-forms W_+^i on U_i $(i = 1, 2)$ which are restricted to $U_1 \cap U_2$ to obey the cocycle condition

$$W_+^1(x) = g_{12}(x) W_+^2 g_{12}^{-1}(x) + g_{12}(x)\, d g_{12}^{-1}(x). \tag{3.3}$$

For each $\lambda \in (0, 1)$ define the connection

$$W_{\lambda+} = (W_{\lambda+}^1, W_{\lambda+}^2) = \left(\mathrm{Im}\, \frac{x\, d\overline{x}}{\lambda^2 + |x|^2}, \mathrm{Im}\, \frac{\lambda^2 \overline{x}\, dx}{|x|^2(\lambda^2 + |x|^2)}\right). \tag{3.4}$$

The connection $W_{\lambda+}$ is self-dual with instanton number one (details on this connection are in [3], [14], and [10]). The curvature of this connection $W_{\lambda+}$ is given by

$$F_{\lambda+} = (F_{\lambda+}^1, F_{\lambda+}^2) = \left(\frac{\lambda^2\, dx \wedge d\overline{x}}{(\lambda^2 + |x|^2)^2}, \frac{\lambda^2 \overline{x}\, dx \wedge d\overline{x} x}{|x|(\lambda^2 + |x|^2)^2 |x|}\right). \tag{3.5}$$

The basic anti-instanton over R^4 is described as follows: The principal $\mathrm{SU}(2)$-bundle over R^4, $\widehat{P}_- \to R^4$, is defined by the transition function

$$g_{12}\colon U_1 \bigcap U_2 \to \mathrm{SU}(2), \qquad g_{12}(x) = \overline{x}/|x|. \tag{3.6}$$

For each $\lambda \in (0, 1)$ define the connection $W_{\lambda-}$ on $\widehat{P}_-$ as

$$W_{\lambda-} = (W_{\lambda-}^1, W_{\lambda-}^2) = \left(\mathrm{Im}\, \frac{\overline{x}\, dx}{x^2 + |x|^2}, \mathrm{Im}\, \frac{\lambda^2 x\, d\overline{x}}{|x|^2(\lambda^2 + |x|^2)}\right). \tag{3.7}$$

The connection $W_{\lambda-}$ is anti-self-dual (with instanton number -1). The curvature of this connection $W_{\lambda-}$ is given by

$$F_{\lambda-} = (F_{\lambda-}^1, F_{\lambda-}^2) = \left(\frac{\lambda^2\, d\overline{x} \wedge dx}{(x^2 + |x|^2)^2}, \frac{\lambda^2 x\, d\overline{x} \wedge dx\overline{x}}{|x|(x^2 + |x|^2)^2 |x|}\right). \tag{3.8}$$

As remarked, we will glue instantons and anti-instantons to a nonminimal solution to the Yang-Mills equations.

First, consider $S^2 \times S^2$; on $S^2 \times S^2$ there exists the complex line bundle

$$L(m, n) = \pi_1^* L^m \otimes \pi_2^* L^n \to S^2 \times S^2$$

for any pair of integers (m, n), where L is the tautological line bundle over S^2, and π_1 and π_2 are the projections from $S^2 \times S^2$ onto its first and second factors. The reducible C^2-vector bundle $L(m, n) \oplus L(m, n)^{-1}$ has second Chern number $-2mn$, and on it sits a reducible SU(2)-connection $A(m, n)$ whose curvature is

$$F_{A(m,n)} = F(m, n) = \frac{1}{2}(m\omega_1 + n\omega_2)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.9}$$

Here $\omega = \pi_1^*\omega$ and $\omega_2 = \pi_2^*\omega$ are the pullbacks of the standard volume form on S^2. As $|m| \neq n|$, $A(m, n)$ is a reducible nonminimal solution to the Yang-Mills equations [24]: If $|m| \neq |n|$ and (m, n) satisfies the conditions

$$\begin{aligned} &\text{if } |m| > |n|, \text{ then } |m| \neq |n|(2k+1) + k(k+1) \quad \text{for } k \geq 0; \\ &\text{if } |n| > |m|, \text{ then } |n| \neq |m|(2k+1) + k(k+1) \quad \text{for } k \geq 0, \end{aligned} \tag{3.10}$$

then $A(m, n)$ is also an isolated solution. This means that the Hessian $\nabla^2 \mathrm{YM}$ at $A(m, n)$ has no null eigenspace on L_{1A}. This connection $A(m, n)$ is discussed in detail in Appendix A and [24]. Let $P(m, n)$ denote the SU(2)-bundle of special unitary frames in $L(m, n) \oplus L(m, n)^{-1}$. In general, we shall only consider a pair of integers (m, n) which satisfies condition (3.10).

With the basic instanton, anti-instanton, and $A(m, n)$ understood, we turn now to the grafting. We will describe this in some generality: Let M be a compact, oriented Riemannian manifold. Let $P_0 \to M$ be a principal SU(2)-bundle and A_0 a smooth connection on P_0 which is an isolated solution of Yang-Mills equations.

To graft basic instantons and anti-instantons onto A_0, we must choose points in M and coordinate systems about the points. A Gaussian coordinate system on a small ball $U(x)$ about $x \in M$ is uniquely specified by a point in the fiber over x of the oriented, orthonormal frame bundle $\pi: F_M \to M$. Indeed, a point $f \in F_M|_x$ identifies $TM|_x$ with $\mathscr{R}^4$. Then the exponential map at x gives a diffeomorphism of a ball in $TM|_x$ with $U(x)$. Together they produce a diffeomorphism $\phi_f: U(x) \to B_\rho \subset \mathscr{R}^4$, the ball of radius ρ. This ϕ_f obeys

$$\begin{aligned} &(1) \quad \phi_f(x) = 0 \text{ and } d\phi_f \cdot f = \{\partial/\partial x^1, \cdots, \partial/\partial x^4\}, \\ &(2) \quad (\phi_f^* dx^\alpha, \phi_f^* dx^\beta) = \delta^{\alpha\beta} + O(|x|^2), \\ &(3) \quad |d(\phi_f^* dx^\alpha, \phi_f^* dx^\beta)| = O(|x|), \end{aligned} \tag{3.11}$$

where (,) is the Riemannian inner product.

A parameter space N of SU(2)-connections over M can be defined as follows.

Definition 3.1. Let $P_0 \to M$ be a principal SU(2)-bundle over M, and let C be a simple, closed geodesic on M. Fix a tubular neighborhood V_0 of C. Let $\{s, y^\alpha\}_{\alpha=1}^3$ be a coordinate system on V_0 with $s: V_0 \to [0, L]$, where $L = \text{length}\, C$. Require that s restrict to C as arclength, and that

$$y^\alpha|_C = 0 \quad \text{and} \quad \{\partial/\partial s, \partial/\partial y^\alpha\}_{\alpha=1}^3$$

be orthonormal on C. Pick up a point $q_0 \in C$. Let ρ be the injectivity radius on M. Fix the integer $k > 0$, and require $d = L/2k < \rho$. Let F_M be the frame bundle over M, and

$$N_0 \subset (d, \rho \times P_0|_{q_0} \times \prod_{i=1}^{2k} \left(\left(0, \frac{r}{2}\right) \times F_M \times \mathrm{SU}(2)\right)$$

be the subset of $(r, g, (\lambda_i, f_i, g_i)_{i=1}^{2k})$ which obeys

$$\text{(3.12)} \qquad \begin{aligned} &(1)\quad d < r < \rho, \\ &(2)\quad 0 < \lambda_i < r/2, \qquad i = 1, \cdots, 2k. \end{aligned}$$

Also, set $q_i = \pi(f_i)$, where π is the projection onto M, and require that

$$(3)\quad 3L/5k > s(q_{i+1}) - s(q_i) > 2L/5k.$$

The set N_0 is a smooth manifold, where $\dim N_0 = 14 \times 2k + 4$.

For each $y \in N_0$, a pair $(P(y), A(y))$ consisting of a principal SU(2)-bundle and the connection $A(y)$ on $P(y)$ will be defined. In the following definition β, $0 \le \beta \le 1$, is a smooth, cut-off function satisfying $\beta(t) = 1$ if $t < 1$ and $\beta(t) = 0$ if $t > 2$. For $1 \le i \le 2k$, the diffeomorphism

$$\phi_i = \phi_{f_i}: U(q_i) \to B_\rho \subset \mathscr{R}^4$$

is a Gaussian coordinate system on a small ball $U(q_i)$. Let A_0 be an SU(2)-connection on P_0, and by means of $g \in P_0|_{q_0}$ fix a gauge along V_0 in which $A_0 = \Gamma + a$, where

$$\text{(3.13)} \qquad \begin{aligned} &(1)\quad \frac{\partial}{\partial s} \lrcorner a = \sigma \text{ which obeys} \\ &(2)\quad \frac{\partial}{\partial s} \lrcorner \nabla_\Gamma \sigma = 0, \\ &(3)\quad a|_C = \sigma|_C\, ds \quad \text{and} \quad y^\alpha \frac{\partial}{\partial y^\alpha} \lrcorner a = 0. \end{aligned}$$

Definition 3.2. Define the family of bundles $(P(y), A(y))$ by the following data:

(1) For each $y = (r, g, (\lambda_i, f_i, g_i)_{i=1}^{2k}) \in N_0$, the cover $\{M\backslash V_0, V_0, V_1, \cdots, V_{2k}, U_1, \cdots, U_{2k}\}$ of M is

$$V_i = \{q \mid \mathrm{dist}(q, q_i) > r\}, \quad 1 \le i \le 2k, \tag{3.14}$$
$$U_i = \{1 \mid \mathrm{dist}(q, q_i) < \lambda_i\} \ 1 \le i \le 2k;$$

(2) The connection $A(y)$ on $P(y)$ has the following expression:

(3.15)

$(\dot{1})$ $A(y) = A_0$ over $M\backslash V_0$;

$(\dot{2})$ $A(y) = \Gamma + \sum_{i=1}^{2k} \beta_r(q - q_i)\phi_i^* g_i W_i^2 g_i^{-1} + a$ over $V_0 \backslash \bigcup_{i=1}^{2k} V_i$;

$(\dot{3})$ $$A(y) = \Gamma + h_i \Big[\phi_i^* g_i W_i^2 g_i^{-1} + (1 - \beta_{\lambda_i}(q - q_i)) \cdot \Big(\sum_{j\neq i} \beta_r(q - q_j)\phi_j^* g_j W_j^2 g_j^{-1} + a\Big)\Big] h_i^{-1} + h_i\, dh_i^{-1}$$
over V_i, $1 \le i \le 2k$;

$(\dot{4})$ $A(y) = \Gamma + \phi_i^* g_i W_i^1 g_i^{-1}$ over U_i, $1 \le i \le 2k$.

Here $(W_i^1, W_i^2) = (W_{\lambda_i +}^1, W_{\lambda_i +}^2)$ if i is odd, and $(W_i^1, W_i^2) = (W_{\lambda_i -}^1, W_{\lambda_i -}^2)$ if i is even. The gauge transformations h_i are given by requiring that $h_i(q_i) = 1$ and that

$$\alpha_i \equiv h_i\Big(a + \sum_{j\neq i} \beta_r(q, q_j)\phi_j^* g_j W_j^2 g_j^{-1}\Big) h_i^{-1} + h_i\, dh_i^{-1} \tag{3.16}$$

obeys

$$\alpha_i(q_i) = 0 \quad \text{and} \quad \frac{\partial}{\partial |q - q_i|} \rfloor \alpha_i = 0. \tag{3.17}$$

Now we give some remarks. First, the pair $(P(y), A(y))$ is smooth. Second, it is obvious that each $A(y)$ is irreducible. Thirdly, by direct calculation, for fixed integer $k > 0$, the bundle $P(y)$, $y \in N_0$, is mutually isomorphic to $C_2(P(y)) = C_2(P_0)$ when $\sup\{\lambda_i \mid 1 \le i \le 2k\}$ is sufficiently small.

Choose a point $y_0 \in N_0$ and write $P = P(y_0)$. For any $y \in N_0$, two isomorphisms $\eta_1, \eta_2 \in \Gamma(\mathrm{Iso}(P, P(y)))$ differ by an element in $\mathrm{Aut}\, P$. Thus, one has the definition of the map $\psi: N_0 \to \mathscr{B}^\natural(P)$.

Definition 3.3. Let N_0 be given as in Definition 3.1. For any $y \in N_0$, pick $\eta(y) \in \gamma(I\,\mathrm{so}(P, P(y)))$ and set

$$\psi(y) = [\eta^*(y)A(y)] \in \mathscr{B}^\sharp(P).$$

Here $P(y)$ and $A(y)$ are given as in Definition 3.2.

A direct calculation shows that ψ is smooth; but it is not injective. The redundant parametrization can be eliminated in the following way: write $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times_{\{\pm 1\}} \mathrm{SU}(2)$. This defines two homomorphisms $\rho_\pm$ of $\mathrm{SO}(4)$ on $\mathrm{SO}(3)$. These representations are mirrored in the geometry with two associated SO(3)-bundles over M, $F_M^\pm \equiv F_M \times \rho_\pm \mathrm{SO}(3)$. (Thus, $F_M^\pm$ is the bundle of oriented, orthonormal frames in $P_\pm \bigwedge^2 T^*M$.)

Definition 3.4. Let k, P_0, A_0, ρ, and V_0 be as in Definition 3.1. We set

$$N_1 \cong (d, \rho) \times \prod_{i=1}^{2k} ((0, r/2) \times F_M^i|_{V_0})/(\Gamma_{A_0} \times \Sigma_k \times \Sigma_k).$$

Here $F_M^i = F_M^\pm$ is the principal SO(3)-bundle of frames in $P_+ \bigwedge^2 T^*M$ (if i is even) or $P_- \bigwedge^2 T^*M$ (if i is odd). Also Γ_{A_0} is isomorphic to the stabilizer of A_0 in the gauge group $\mathrm{Aut}(P_0)$, and Σ_k = symmetric group on k-letters. In the quotient

$$\prod_{i=1}^{2k} ((0, r/2) \times F_M^i|_{V_0})/\Gamma_{A_0},$$

the group Γ_{A_0} acts diagonally on $\prod_{i=1}^{2k} F_M^i|_{V_0}$.

The next proposition is analogous to Proposition 4.5 in [19].

Proposition 3.1. *The map* $\psi\colon N_0 \to \mathscr{B}^\sharp(P)$ *of Definition* 3.3 *factors through* N_1.

Proof. The proof mimics the proof of the Proposition 4.5 in [21]. Without loss of generality, restrict to the ball

$$V_1 = \{q \in M \mid \mathrm{dist}(q, q_1) < r\}$$

centered at $q_1 = \pi(f_1)$. Let

$$y = (r, g, (\lambda_1, f_1, g_1), (\lambda_i, f_i, g_i)_{i=2}^{2k})$$

and

$$y' = (r, g, (\lambda_1, f_1 e, \tilde{g} g_1), (\lambda_i, f_i, g_i)_{i=2}^{2k}),$$

where $\tilde{g} \in \mathrm{SU}(2)$, and $e = [e^+, e^-] \in \mathrm{SO}(4) \cong \mathrm{SU}(2) \times_{\{\pm\}} \mathrm{SU}(2)$. Let $\{x^\alpha\}_{\alpha=1}^4$ and $\{x_e^\alpha\}_{\alpha=1}^4$ denote the Gaussian coordinate systems defined by

f_1 and $f_1 e$ respectively. By thinking of $\mathscr{R}^4 = \mathscr{H}$ (quaternions) and $\mathrm{SU}(2) \cong S^3 \subset \mathscr{H}$, we have

$$X_e = e_+ X e_-^{-1}. \tag{3.18}$$

Hence the transition function φ in $V_1 \cap \{M \setminus \bigcup_{i=1}^{2k} \pi(F_i)\}$ defined by f_1 and $f_1 e$ are related by

$$\varphi[f_1 e] = \phi_{f_1 e}^*(x/|x|) = e_+ \varphi[f_1] e_-^{-1}. \tag{3.19}$$

Since $(W_1^1, W_1^2) = (W_{\lambda_1 +}^1, W_{\lambda_1 +}^2)$, the connection 1-form α_1 is

(3.20)

$$\begin{aligned} &\phi_{f_1 e}^* \tilde{g} g_1 W_1^2 g_1^{-1} \tilde{g}^{-1} + (1 - \beta_{\lambda_1}) \left(a + \sum_{j \neq 1} \beta_r \phi_{f_j}^* g_j W_j^2 g_j^{-1} \right) \\ &\quad = e_- \tilde{g} (\phi_{f_1}^* g_1 W_1^2 g_1^{-1}) \tilde{g}^{-1} e_-^{-1} + (1 - \beta_{\lambda_1}) \left(a + \sum_{j \neq 1} \beta_r \phi_{f_j}^* g_j W_j^2 g_j^{-1} \right). \end{aligned}$$

One concludes from (3.11)–(3.15) that the images of

$$y = (r, g, (\lambda_1, f_1, g_1), (\lambda_i, f_i, g_i)_{i=2}^{2k})$$

and

$$y' = (r, g, (\lambda_1, f_1 e, \tilde{g} g_1), (\lambda_i, f_i, g_i)_{i=2}^{2k})$$

in $\mathscr{B}^\sharp$ coincide when $\tilde{g}^{-1} = e_-$. Since a permutation of the factors (λ_i, f_i, g_i), i is odd (or even), of y changes nothing, the map ψ is equivariant under the symmetric group $\Sigma_k \times \Sigma_k$.

Next, since Γ_{A_0} is the centralizer of the gauge group $\mathrm{Aut}(P_0)$, the group Γ_{A_0} acts diagonally on $\prod_{i=1}^{2k} F_M^i$. On the other hand, $A_0|_{V_0} = \Gamma + a$; this gauge is unique up to $a \to hah^{-1}$ for $h \in \mathrm{SU}(2)$. For any $\tilde{g} \in \Gamma_{A_0}$, $a(g\tilde{g}^{-1}) = \tilde{g}^{-1}$ and $a(g)\tilde{g} = a(g)$. Therefore the map ψ is equivariant under the Γ_{A_0} acting diagonally on $\prod_{i=1}^{2k} F_M^i$. Hence ψ factors through N_1. q.e.d.

As a corollary, ψ maps

$$\overline{N}_1 \subset (d, \rho) \times \prod_{i=1}^{2k} ((0, r/2) \times F_M^i)/\Sigma_k \times \Sigma_k$$

into $\mathscr{B}'(P)$. If A_0 is reducible, then $\Gamma_{A_0} = \mathrm{U}(1)$ and $\overline{N}_1$ is U(1)-bundle over N_1.

For convenience of later calculations, a change of parameters is useful.

Definition 3.5. As in Definition 3.4, for a sufficiently large positive integer $k > 0$, define N_2 to be the following subset:

$$N_2 \subset (d, \rho) \times \prod_{i=1}^{2k} ((0, r/2) \times F_M)/\Gamma_{A_0} \times \Sigma_k \times \Sigma_k .$$

A point $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in N_2$ exists if

(1) $r = d^{3/5}$, $\lambda_i = s_i d^2$, $1 \le i \le 2k$, and

(2) $(r, (\lambda_i, f_i, g_i)_{i=1}^{2k}) \in N_0$.

It is easy to check that N_2 is a smooth manifold. The induced map from N_2 into $\mathscr{B}^\sharp$ will still be denoted by ψ.

Proposition 3.2. *If a positive integer k is large, then the map $\psi: N_2 \to \mathscr{B}^\sharp$ is an embedding.*

Proof. Propositions 4.2 and 4.4 show that ψ is an immersion. To prove that ψ is 1-1, look first at the points in M where the curvature form $|F_{A(y)}|$ has a local maximum. Look also at the values of $|F_{A(y)}|$ at these points. If ψ maps y and y' to the same orbit in $\mathscr{B}^\sharp$, then this curvature information implies that, up to the action of $\Sigma_k \times \Sigma_k$, $s_i = s_i'$ and $\pi(f_i) = \pi(f_i')$ for all i. With this understood, one can go back to (3.14)–(3.15) to readily show that ψ is globally 1-1. q.e.d.

Using the map ψ the manifold N_2 is the parameter space for our approximate solutions to the Yang-Mills connections over M.

Proposition 3.3. *Suppose that $y = ((s_i, f_i, g_i)_{i=1}^{2k})$ is in N_2. Then the corresponding* SU(2)*-connections $A(y)$ on P over M has*

$$\|\nabla \mathrm{YM}_{A(y)}(\cdot)\|^*_{A(y)} \le C \left\{ \sum_{i=1}^{2k} s_i^2 d^{26/5} \right\}^{1/2}, \tag{3.21}$$

where C is a constant which is independent of y and d (or $k(d = \pi/k)$).

Proof. By a direct calculation, as in Appendix B. q.e.d.

Now, returning to $S^2 \times S^2$, let C be a closed geodesic on the first S^2. Choose the pair $(P(m, n), A(n, m))$ as the background SU(2)-bundle and connection. With this done, we now have a prescription for an approximate solution space for the Yang-Mills equations.

As for $S^1 \times S^3$, take the product metric on $S^1 \times S^3$. Then $S^1 \equiv S^1 \times \{\mathrm{pt}\}$ is a closed geodesic in $S^1 \times S^3$. In Appendix A we show that the Levi-Civita connection on TS^3 defines an irreducible nonminimal Yang-Mills SU(2)-connection A_0 with degree zero over $S^1 \times S^3$. Furthermore, it is an isolated solution to the Yang-Mills equations.

A family of approximate solutions to the Yang-Mills equations on $S^2 \times S^2$ (or $S^1 \times S^3$) has now been constructed. Our job is to solve the Yang-Mills equations near these approximate solutions. As indicated, we can solve the Yang-Mills equations in orthogonal direction to the small eigenvalues of $\nabla^2 \mathrm{YM}_{A(y)}$ on $L_{\nu 1 A(y)}$. If the positive integer k is large enough, perturbation arguments as in §§7–8 of [21] allow one to prove the following:

Proposition 3.4. *Let $M = S^2 \times S^2$ or $S^1 \times S^3$. Then there exists $\varepsilon_0 > 0$ so that given $\varepsilon_0 > \varepsilon > 0$ there exists $0 < k < \infty$ such that if $\xi \in (\varepsilon, 2\varepsilon)$, then ξ is not in the spectrum of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ for all $y \in N_2$. Also, for all $y \in N_2$,*

$$\mathrm{Spec}(\nabla^*_{A(y)} \nabla_{A(y)}) \bigcap (\varepsilon, 2\varepsilon) = \varnothing .$$

For such ξ and $\nu \equiv 3\varepsilon/2$, there exists

$$\mathscr{U}_\xi(y) \in \{(1 - \pi_\nu(A(y), \xi)) + \pi_\nu(A(y), -\xi)\} L_{\nu 1 A(y)} \bigcap \Gamma(\mathrm{Ad}\, P \otimes T^* M)$$

such that

$$\begin{aligned} \{(1 - \pi_\nu(A(y) + \mathscr{U}_\xi(y), \xi)) + \pi_\nu(A(y) + \mathscr{U}_\xi(y), -\xi)\} \\ \cdot \nabla \mathrm{YM}_{A(y) + \mathscr{U}_\xi(y)}(\cdot) = 0 , \end{aligned} \tag{3.22}$$

$$\begin{aligned} \|\mathscr{U}_\xi(y)\|_{A(y)} \leq C \|\{(1 - \pi_\nu(A(y), \xi)) + \pi_\nu(A(y), -\xi)\} \\ \cdot \nabla \mathrm{YM}_{A(y)}(\cdot)\|^*_{A(y)} , \end{aligned} \tag{3.23}$$

where C depends only on ν, $\mu_\nu[A(y)]$, and $\Delta_{\nu\xi[A(y)]}$.

One last comment: $A(y) + \mathscr{U}_\xi(y)$ is also an irreducible SU(2)-connection according to the argument in [17, §8].

4. The obstruction

In the last section we constructed the parameter spaces N_2 of approximate solutions to the Yang-Mills equations. We would like to solve the Yang-Mills equations near the approximate solutions by a Lyapunov-Schmidt method. The small eigenvalues of the Hessian of the Yang-Mills functional are the obstruction to solving the Yang-Mills equations. So, in this section, the goal is to study the small eigenvalues of the Hessian of YM at an approximation solution, and analyze the obstruction by the small eigenvalues.

According to the construction in §3, for any $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in N_2$, there exists a smooth, irreducible SU(2)-connection $A(y)$ over $S^2 \times S^2$, and the map $\psi: N_2 \to \mathscr{B}^\sharp$ is an embedding (if k is enough large). $A(y)$ is an approximate solution to the Yang-Mills equations on $S^2 \times S^2$. As the Hessian $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ of YM at $A(y)$ is restricted to $L_{\nu 1 A(y)}$, the spectrum of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ in the interval $(-\infty, 1)$ is pure point spectrum, the eigenvalues in $(-\infty, 1)$ have finite multiplicities, and their only accumulation point in $(-\infty, 1)$ is the number 1. Likewise, for $a \in L_1^2(\Omega^1(\mathrm{Ad}\, P))$, if $\|a\|_{A(y)} < \varepsilon$, then the Hessian $\nabla^2 \mathrm{YM}_{A(y)+a}$ defines a closed, symmetric bilinear form on $L_{\nu 1 A(y)+a}$ with discrete spectrum in $(-\infty, 1)$, which has no accumulation points. Since our approximate solutions are smooth, elliptic regularity theory insures that the eigenvectors of $\nabla^2 \mathrm{YM}$ are all smooth sections on $\mathrm{Ad}\, P \otimes T^*(S^2 \times S^2)$. For the details, readers are referred to [21, §§7–9].

To study the obstruction to solving the Yang-Mills equations, we need to study the spectrum of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$. Recall the construction. We take the reducible SU(2)-connection $A(m, n)$ as background connection; it is an isolated solution to the Yang-Mills equations on $S^2 \times S^2$. Setting $\nu = 0$, we see that

$$\mathscr{G}_0[A(m, n)] = \{\sigma \in L_1^2(\mathrm{Ad}\, P) \mid \nabla^*_{A(m, n)} \nabla_{A(m, n)} \sigma = 0\}$$

is isomorphic to the real line R. $\mu_0[A(m, n)]$ is a positive constant. As $\nabla^2 \mathrm{YM}_{A(m, n)}$ is restricted to $L_{01A(m, n)}$, then $\nabla^2 \mathrm{YM}_{A(m, n)}$ on $L_{01A(m, n)}$ has discrete spectrum in $(-\infty, 1)$. We let $\xi_1' < \xi_2' < \cdots \xi_n' < \cdots$ denote the eigenvalues of $\nabla^2 \mathrm{YM}_{A(m, n)}$, where $\xi_i' \neq 0$ for $i = 1, 2, \cdots$, and the dimension $k(m, n)$ of the negative eigenspace of $\nabla^2 \mathrm{YM}_{A(m, n)}$ is finite. According to Lemma 6.8 in §6 of [21], as the parameter k is sufficiently large and $a \in L_1^2(\mathrm{Ad}\, P \otimes T^*(S^2 \times S^2))$ such that $\|a\|_{A(y)}$ is sufficiently small, then $\nabla^*_{A(y)} \nabla_{A(y)}$ (or $\nabla^*_{A(y)+a} \nabla_{A(y)+a}$) has $\dim \mathscr{G}_0[A(m, n)]$ eigenvectors with eigenvalues in $[0, \mu_0[A(m, n)]/4)$, and all other eigenvalues of $\nabla^*_{A(y)} \nabla_{A(y)}$ (or $\nabla^*_{A(y)+a} \nabla_{A(y)+a}$) on $L_2^2(\mathrm{Ad})$ are in the interval $[3\mu_0[A(m, n)]/4, \infty)$. Hence, we take $\nu_0 = 1/4\mu_0[A(m, n)]$, and the Hessian $\nabla^2 \mathrm{YM}_{A(y)}$ (or $\nabla^2 \mathrm{YM}_{A(y)+a}$) is restricted to $L_{\nu 1 A(y)}$ (or $L_{\nu 1 A(y)+a}$).

Now we turn our attention to the Hessian $\nabla^2 \mathrm{YM}$ of the functional YM for the approximate solutions. The approximate solutions are composed from the background connection (a reducible SU(2)-connection, which is an isolated solution to the Yang-Mills equations) and the standard self-dual and anti-self-dual SU(2)-connections over $\mathscr{R}^4$; which are grafted onto the background connection by the gluing operation. The small eigenvalues of the Hessian are determined by the eigenvalues of the Hessian of YM for the standard self-dual and anti-self-dual SU(2)-connections over $\mathscr{R}^4$.

According to the construction of approximate solutions, it is reasonable to conjecture that the small eigenvalues of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ come from the small eigenvalues of $\nabla^2 \mathrm{YM}_{A(m,n)}(\cdot, \cdot)$ and $\nabla^2(\mathrm{YM}_{W\pm}(\cdot, \cdot)$. In fact, C. H. Taubes has established this property in §7 of [21]. Let $n_0(\xi, \nu_0)$ denote the number of eigenvectors of $\nabla^2 \mathrm{YM}_{A(m,n)}(\cdot, \cdot)$ on $L_{\nu_0 1 A(m,n)}$ with eigenvalues less than ξ. Let $n(W, \xi)$ denote the number of eigenvectors of $\nabla^2 \mathrm{YM}_{W_\pm}(\cdot, \cdot)$ on $L_{\mathscr{R}^4, W_\pm}$ with eigenvalues less than ξ. For any $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in N_2$, set $n(A(y), \xi) = n_0(\xi, \nu_0) + 2kn(W, \xi)$.

The following proposition is not proved here; for the details, readers are referred to [21, §7].

Proposition 4.1. *For $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in N_2$, let $A(y)$ be an approximate solution to the Yang-Mills equations. Fix $\xi < 1$, and suppose that $\Delta_{\nu_0 \xi}[A(y)] > 2\delta$. Then as d is very small (or k is very large), the following hold*:

(1) *The number of eigenvectors of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ on $L_{\nu_0 1 A(y)}$ with eigenvalues less than $\xi + \delta$ is not less than $n(A(y), \xi)$.*

(2) *The number of eigenvectors of $\nabla^2 M_{A(y)}(\cdot, \cdot)$ on $L_{\nu_0 1 A(y)}$ with eigenvalues less than $\xi + \Delta_{\nu_0 \xi}[A(y)] - \delta$ is not greater than $n(A(y), \xi)$.*

In this and the remaining sections fix $\xi_0 = \frac{1}{4}\delta_{\nu_0 0}[A(y)]$. There is an analogous conclusion for $S^1 \times S^3$; the details are omitted.

Set $N = \psi(\overline{N}_2)$. To analyze the obstruction, we now study the tangent space TN of N, and recall the construction in §3. For $y \in N_2$, $A(y) = \psi(y) \in \mathscr{B}^\natural$, and $\overline{N}_2 \to N_2$ is a principal U(1)-bundle over N_2, $\overline{N}_2 \subset \mathscr{B}'$. For later use, we take the following gauge equivalent class of $A(y)$:

(i̇) $A(y) = A(m, n)$ over $S^2 \times S^2 \backslash V_0$;

(2̇)
$$A(y) = \Gamma + \prod_{i=1}^{2k}(1 - \beta_{S_i d^2}(q - q_i))a + \sum_{i=1}^{2k}\prod_{j \neq i}(1 - \beta_{S_j d^2}(q - q_j))\beta_r(q - q_i)\phi_i^* g_i W_i^2 g_i^{-1} \quad \text{over } V_0 \backslash \bigcup_{i=1}^{2k} U_i;$$

$$(\dot{3})\quad A(y) = \Gamma + \phi_i^* g_i W_i^1 g_i^{-1} \quad \text{over } U_i, \ 1 \le i \le 2k.$$

Let $\{T^\alpha\}_{\alpha=1}^3$ be the orthonormal basis of the Lie algebra of SU(2). Likewise, for $q_i \in S^2 \times S^2$, $1 \le i \le 2k$, let $\{x_i^t\}_{t=1}^4$ denote the local coordinate systems of the neighborhoods of q_i. For each $y \in \overline{N}_2(s, f, g)$, the tangent space $T_y\overline{N}_2(s, f, g)$ is spanned by $\{\partial/\partial s_i\}_{1\le i\le 2k}$, $(\{T_i^2\}_{\alpha=1}^3)_{1\le i<2k}$, and $(\{\partial/\partial x_i^t\}_{t=1}^4)_{1\le i\le 2k}$, where $\{\partial/\partial x_i^1, \partial/\partial x_i^2\}_{1\le i\le 2k}$ and $\{\partial/\partial x_i^3, \partial/\partial x_i^4\}_{1\le i\le 2k}$ are orthonormal bases of the tangent spaces $\pi_1^* TS^2$ and $\pi_2^* TS^2$ respectively, and π_1, π_2 are projections to two factors of $S^2 \times S^2$.

By direct calculation, we shall find that TN is close to the small eigenvectors of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ for each $A(y) \in N$. In fact, for each $y \in \overline{N}_2$, the tangent space $T_{(A(y))}N$ modulo $D_{A(y)}\Gamma(\mathrm{Ad}\, P)$ can be written as follows:

(1) on $B_{2r}(q_i)$

$$\psi_*\left(\frac{\partial}{\partial s_i}\right) = -\frac{\partial \beta_{s_i d^2}(x - q_i)}{\partial s_i} a + \prod_{j\ne i}(1 - \beta_{s_j d^2}(x - q_j))\beta_r(x - q_i)\phi_i^* g_i \frac{\partial W_i^2}{\partial s_i} g_i^{-1},$$

on $B_{2s_i d^2}(q_i)$

$$\psi_*\left(\frac{\partial}{\partial s_i}\right) = \phi_i^* g_i \frac{\partial W_i^1}{\partial s_i} g_i^{-1}, \qquad 1 \le i \le 2k;$$

(2) on $B_{2r}(q_i)$.

$$\psi_*(T_k^\alpha) = \prod_{j\ne i}(1 - \beta_{s_j d^2}(x - q_j))\beta_r(x - q_i)[T_i^\alpha, \phi_i^* g_i W_i^2 g_i^{-1}],$$

on $B_{2s_i d^2}(q_i)$

$$\psi_*(T_k^\alpha) = [T_i^\alpha, \phi_i^* g_i W_i^1 g_i^{-1}], \qquad \alpha = 1, 2, 3, \ 1 \le \alpha \le 2k;$$

(3) on $B_{2r}(q_i)$

$$\psi\left(\frac{\partial}{\partial x_i^t}\right) = -\frac{\partial \beta_{s_i d^2}(x-q_i)}{\partial x_i^t} a$$
$$+\prod_{j\neq i}(1-\beta_{s_j d^2}(x-q_j))\frac{\partial \beta_r(x-q_i)}{\partial x_i^t}\phi_i^* g_i W_i^2 g_i^{-1}$$
$$+\prod_{j\neq i}(1-\beta_{s_j d^2}(x-q_j))\beta_r(x-q_i)\phi_i^* g_i \frac{\partial W_i^2}{\partial x_i^t} g_i^{-1},$$

on $B_{2s_i d^2}(q_i)$

$$\psi_*\left(\partial/\partial x_i^t\right) = \phi_i^* g_i \frac{\partial W_i^1}{\partial x_i^t} g_i^{-1}, \qquad t = 1, 2, 3, 4, \ 1 \leq i \leq 2k.$$

Here $r = d^{3/5}$, $d = \pi/k$. Now we have the following estimates.

Proposition 4.2. *Fix a parameter $k > 0$. Let $d = \pi/k$ be sufficiently small and suppose $y \in \overline{N}_2$. Then*

$$\left\{\psi_*\left(\frac{\partial}{\partial x_i^t}\right)_{1\leq t\leq 4}, \psi_*\left(\frac{\partial}{\partial s_i}\right), \psi_*(T_i^\alpha)_{1\leq\alpha\leq 3}\right\}_{i=1}^{2k}$$

are linearly independent, and we have

(1) $$\left\|D_{A(y)}^*\psi_*\left(\frac{\partial}{\partial s_i}\right)\right\|_{L_2} + \operatorname*{Sup}_{q\in S^2\times S^2}\left\|\operatorname{dist}(q, q_i)^{-2} D_{A(y)}^*\psi_*\left(\frac{\partial}{\partial s_i}\right)\right\|_{L^1} \leq Z s_i d^2 \left\|\psi_*\left(\frac{\partial}{\partial s_i}\right)\right\|_{A(y)} \quad \textit{for } 1 \leq i \leq 2k;$$

(2) $$\|D_{A(y)}^*\psi_* T_i^\alpha)\|_{L^2} + \operatorname*{Sup}_{q\in S^2\times S^2}\left\|\operatorname{dist}(q, q_i)^{-2} D_{A(y)}^*\psi_*\left(\frac{\partial}{\partial s_i}\right)\right\|_{L^1} \leq Z s_i d^2 \|\psi_*(T_k^\alpha)\|_{A(y)} \quad \textit{for } 1 \leq \alpha \leq 3, \ 1 \leq i \leq 2k;$$

(3) $$\left\|D_{A(y)}^*\left(\frac{\partial}{\partial x_i^t}\right)\right\|_{L^2} + \operatorname*{Sup}_{q\in S^2\times S^2}\left\|\operatorname{dist}(q, q_i)^{-2} D_{A(y)}^*\psi_*\left(\frac{\partial}{\partial x_i^t}\right)\right\|_{L^1} \leq Z s_i d^2 \left\|\psi_*\left(\frac{\partial}{\partial x_i^t}\right)\right\|_{A(y)} \quad \textit{for } 1 \leq t \leq 4, \ 1 \leq i \leq 2k$$

where Z is a positive constant.

Proof. This is done by direct calculation from the formulas.

Proposition 4.3. *Suppose* $y \in \overline{N}_2$. *Then for any* $q \in S^2 \times S^2$ *obeying* $\operatorname{dist}(q, q_i) > 16 s_i d^2$, *we have*

(1) $$\left|\nabla^{(l)}_{A(y)} \psi_* \left(\frac{\partial}{\partial s_i}\right)\right|(q) \le C_l s_i^2 d^4 \operatorname{dist}(q, q_i)^{-l-3} \left\|\psi_* \left(\frac{\partial}{\partial s_i}\right)\right\|_{A(y)}$$ *for* $1 \le i \le 2k$;

(2) $$|\nabla^{(l)}_{A(y)} \psi_*(T^\alpha_k)|(q) \le C_l s_i^2 d^4 \operatorname{dist}(q, q_i)^{-l-3} \|\psi_*(T^\alpha_k)\|_{A(y)}$$ *for* $1 \le \alpha \le 3$, $1 \le i \le 2k$;

(3) $$\left|\nabla^{(l)}_{A(y)} \psi_* \left(\frac{\partial}{\partial x^t_i}\right)\right|(q) \le C_l s_i^2 d^4 \operatorname{dist}(q, q_i)^{-l-3} \left\|\psi_* \left(\frac{\partial}{\partial x^t_i}\right)\right\|_{A(y)}$$ *for* $1 \le t \le 4$, $1 \le i \le 2k$,

where C_l *are positive constants.*

Proof. By direct calculation. q.e.d.

As in [21, §7], we may define the map $J: TN \to L_{\nu_0 1 A(y)}$. Suppose that $y \in \overline{N}_2$. For any $v \in T_{A(y)}N$, define

$$J(v) = v + \nabla_{A(y)}\sigma, \tag{4.1}$$

where σ satisfies the conditions

$$(1 - \pi_{\nu_0}(A(y)))\sigma = \sigma \tag{4.2}$$

and

$$(1 - \pi_{\nu_0}(A(y)))\{\nabla^*_{A(y)} v + \nabla^*_{A(y)} \nabla_{A(y)}\sigma\} = 0. \tag{4.3}$$

Let π_y denote the projection of $J(TN)$ onto $\Omega(y)$, where

$$\Omega(y) = (1 - \pi(y, -\xi_0)) \circ \pi(y, \xi_0) L_{\nu_0 1 A(y)}$$

is the space of small eigenvectors of $\nabla^2 \mathrm{YM}_{A(y)}(\cdot, \cdot)$ on $L_{\nabla_0 | A(y)}$.

The following proposition is a consequence of Propositions 4.1–4.3.

Proposition 4.4. *Let* $d = \pi/k$ *be sufficiently small. Suppose that* $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in \overline{N}_2$. *Then the space* $\Omega(y)$ *of small eigenvectors of* $\nabla^2(\mathrm{YM}_{A(y)}(\cdot, \cdot)$ *on* $L_{\nabla_0 | A(y)}$ *is isomorphic to the tangent space* $T_{A(y)}N$. *Furthermore, for each* $v \in T_{A(y)}N$

$$\begin{aligned} |\nabla^2 \mathrm{YM}_{A(y)}(v, v)| &\le Z \sum_{i=1}^{2k} s_i d^2 \|v\|^2_{A(y)}, \\ \|\pi_y \cdot J(v) - v\|_{A(y)} &\le Z \sum_{i=1}^{2k} s_i d^2 \|v\|_{A(y)}. \end{aligned} \tag{4.4}$$

Notice $(\pi_y \circ J(\psi_*(\partial/\partial s_i)))_{1\leq i\leq 2k}$, $(\{\pi_y \circ J(\psi_*(T_i^\alpha))\}_{\alpha=1}^3)_{1\leq i\leq 2k}$, and $(\{\pi_y \circ J(\psi_*(\partial/\partial x_i^t))\}_{t=1}^4)_{1\leq i\leq 2k}$ belong to

$$\Omega(y) \cap C^\infty(S^2 \times S^2, \operatorname{Ad} P \otimes T^* S^2 \times S^2).$$

Since, for any $g \in \operatorname{Aut} P$,

$$L_{\nu_0 1 g\cdot A(y)} = g L_{\nu_0 1 A(y)}, \qquad \mathcal{G}_{\nu_0}[gA(y)] = g\mathcal{G}_0[A(y)]$$

and $\overline{N}_2 \to N_2$ is the U(1)-bundle, the map $\pi_y \circ J \colon T_{A(y)} N \to \Omega(y)$ is U(1)-equivariant.

Our goal is to solve (2.16), which provides a map $N \to L^*_{\nu_0 1 A(y)}$. For each $v \in T_y \overline{N}_2$, set $\overline{\psi}_*(v) = \psi_*(v)/\|\psi_*(v)\|_{A(y)}$. Define the map $f\colon \overline{N}_2 \to \Omega(y)$ as follows: For each $y \in \overline{N}_2$

$$\begin{aligned} f(y) = \sum_{i=1}^{2k} \Bigg\{ &\left[\nabla M_{A(y)}\left(\pi_y \circ \left(\psi_*\left(\frac{\partial}{\partial s_i}\right)\right)\right)\right. \\ &\left. + R\left(A(y), \xi_0, \mathcal{U}_{\xi_0}(y); \pi_y \circ J\left(\psi_*\left(\frac{\pi}{\partial s_i}\right)\right)\right)\right] \\ &\cdot \pi_y \circ J\left(\psi_*\left(\frac{\partial}{\partial s_i}\right)\right) \\ &+ \sum_{\alpha=1}^{3} [\nabla \operatorname{YM}_{A(y)}(\pi_y \circ J(\psi_*(T_i^\alpha))) \\ &+ R(A(y), \xi_0, \mathcal{U}_{\xi_0}(y); \pi_y \circ J(\psi_*(T_i^\alpha)))]\pi_y \circ J(\psi_*(T_i^\alpha)) \\ &+ \sum_{t=1}^{4} \left[\nabla \operatorname{YM}_{A(y)}\left(\pi_y \circ J\left(\psi_*\left(\frac{\partial}{\partial x_i^t}\right)\right)\right)\right. \\ &\left. + R\left(A(y), \xi_0, \mathcal{U}_{\xi_0}(y)\ ;\ \pi_y \circ J\left(\psi_*\left(\frac{\partial}{\partial x_i^t}\right)\right)\right)\right] \pi_y \\ &\circ J\left(\psi_*\left(\frac{\partial}{\partial x_i^t}\right)\right)\Bigg\}. \end{aligned} \tag{4.5}$$

As $\nabla \mathrm{YM}$ is invariant under the action of $\mathrm{Aut}\, P$, and $\mathscr{U}_{\xi_0}$ is an an $\mathrm{Aut}\, P$-equivariant map, f is viewed as $\mathrm{U}(1)$-equivariant map from $\overline{N}_2$ to $\Omega(y)$. Hence, the problem at hand is to determine under what circumstances the $\mathrm{U}(1)$-equivariant map f has a zero. This task is completed in §5.

There is one more useful estimate needed:

Proposition 4.5. *Assume that $d = \pi/k$ is small, and suppose that $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in \overline{N}_2$. Then for each $v \in T_y\overline{N}_2$ we have the following a priori estimates*:

$$(4.6) \qquad 0 < C_1 \le \left\| \psi_* \left(\frac{\partial}{\partial s_i} \right) \right\|_{A(y)} \le C_2 < \infty, \qquad 1 \le i \le 2k,$$

$$(4.7) \qquad 0 < C_1 \le \| \psi_* (T_i^\alpha) \|_{A(y)} \le C_2 < \infty, \qquad 1 \le \alpha \le 3, \ 1 \le i \le 2k,$$

$$(4.8) \qquad 0 < C_1 \le \left\| \psi_* \left(\frac{\partial}{\partial x_i^t} \right) \right\|_{A(y)} \le C_2 < \infty, \qquad 1 \le t \le 4, \ 1 \le i \le 2k,$$

where C_1 and C_2 are positive constants which depend only on the parameter s_i.

Proof. This is another direct calculation. q.e.d.

In the next section we shall use these estimates to find positions of the parameters that make $f(y) = 0$; that is, to find $y \in N_2$ such that

$$(4.9) \qquad \begin{aligned} &(1 - \pi(y, -\xi_0)^*) \\ &\quad \circ \pi(y, \xi_0)^* \{ \nabla \mathrm{YM}_{A(y)}(\cdot) + R(A(y), \xi_0, \mathscr{U}_{\xi_0}(y); \cdot) \} = 0. \end{aligned}$$

5. The proofs of Theorems 1.1 and 1.2

In this section we shall complete the proofs of Theorems 1.1 and 1.2. In the last section, for $S^2 \times S^2$, we constructed a $\mathrm{U}(1)$-equivariant map $f\colon N_2 \to \Omega(y)$. Our method for solving equation (4.9) will be to decompose f into $f^1 + f^2$ and so reduce (4.9) to the equation for a critical point for the functional $\mathrm{YM}(A(y) + \mathscr{U}_{\xi_0}(y))$ on the parameter space N_2.

For this purpose, we now study the Taylor's expansion of the Yang-Mills action $\mathrm{YM}(A(y) + \mathscr{U}_{\xi_0}(y))$ in parameters s_i and d for each $y \in N_2$. Using the a priori estimates for $\mathscr{U}_{\xi_0}(y)$ which are given in §3, we can derive the

following expansion (see Appendix B):
(5.1)

$$
\begin{aligned}
\mathrm{YM}(A(y)+\mathscr{U}_{\xi_0}(y)) &= 8\pi^2(m^2+n^2+2k) \\
&+\sum_{\text{odd } i} -\frac{\omega(S^3)}{2} s_i^2 d^4 \langle P_- F(m,n)(q_i), \phi_i^* g_i F_-(N) g_i^{-1}\rangle \\
&+\sum_{\text{even } j} -\frac{\omega(S^3)}{2} s_j^2 d^4 \langle P_+ F(m,n)(q_j), \phi_j^* g_j F_+(N) g_j^{-1}\rangle \\
&+\sum_{\text{odd } i} \sum_{\substack{\text{even } j \\ j=[i\pm(2l+1)] \bmod 2k \\ 0\le l\le [d^{-2/5}/2]-1}} -Q\frac{\omega(S^3)}{2}\cdot\frac{s_i^2 s_j^2 d^8}{\operatorname{dist}(q_i,q_j)^4} \\
&\quad \cdot \langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1}\rangle \\
&+\sum_{\text{even } j} \sum_{\substack{\text{odd } i \\ i=[j\pm(2l+1)] \bmod 2k \\ 0\le l\le [d^{-2/5}/2]-1}} -Q\frac{\omega(S^3)}{2}\cdot\frac{s_j^2 s_i^2 d^8}{\operatorname{dist}(q_j,q_i)^4} \\
&\quad \cdot \langle \phi_j^* g_j F_+(N) g_j^{-1}, \phi_i^* g_i F_+(N) g_i^{-1}\rangle \\
&+\sum_{i=1}^{2k}\{s_k^2 d^{26/5}(C_1+C_2|\ln d|)+\text{higher order terms}\},
\end{aligned}
$$

where Q is a positive constant which is independent of our parameters. We divide $\mathrm{YM}(A(y)+\mathscr{U}_{\xi_0}(y))$ into two parts:

$$
\mathrm{YM}(A(y)+\mathscr{U}_{\xi_0}(y)) = H_1(y)+H_2(y), \tag{5.2}
$$

where

$$
\begin{aligned}
H_2(y) &= \sum_{i=1}^{2k}\{s_i^2 d^{26/5}(C_1+C_2|\ln d|)+\text{higher order terms}\}, \\
H_1(y) &= \mathrm{YM}(A(y)+\mathscr{U}_{\xi_0}(y)-H_2(y).
\end{aligned} \tag{5.3}
$$

For any $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in \overline{N}_2$, define

(5.4)

$$f^1(y) = \sum_{i=1}^{2k} \left\{ \frac{\partial}{\partial s_i} H_1(y) \left\| \psi_* \left(\frac{\partial}{\partial s_i} \right) \right\|_{A(y)} \cdot \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial s_i} \right) \right.$$
$$+ \sum_{\alpha=1}^{3} T_i^\alpha H_1(y) \|\psi_*(T_i^\alpha)\|_{A(y)} \cdot \pi_y \circ J \circ \overline{\psi}_*(T_i^\alpha)$$
$$\left. + \sum_{t=1}^{4} \frac{\partial}{\partial x_i^t} H_1(y) \left\| \psi_* \left(\frac{\partial}{\partial x_i^t} \right) \right\|_{A(y)} \cdot \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial x_i^t} \right) \right\}$$

$$= \sum_{i \text{ odd}} \left\{ -\omega(S^3) s_i d^4 \langle P\_F(m, n)(q_i), \phi_i^* g_i F_-(N) g_i^{-1} \rangle \right.$$
$$+ \sum_{\substack{\text{even } j \\ j=[i\pm(2l+1)] \mod 2k \\ 0 \le l \le [d^{-2/5}/2]-1}} -Q\omega(S^3) \frac{s_i^1 s_j^2 d^8}{\operatorname{dist}(q_i, q_j)^4}$$
$$\cdot [\langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1} \rangle$$
$$\left. + \langle \phi_j^* g_j F_+(N) g_j^{-1}, \phi_i^* g_i F_+(N) g_i^{-1} \rangle] \right\}$$
$$\cdot \left\| \psi_* \left(\frac{\partial}{\partial s_i} \right) \right\|_{A(y)} \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial s_i} \right)$$
$$+ \sum_{\text{even } j} \left\{ -\omega(S^3) s_j d^4 \langle P_+ F(m, n), \phi_j^* g_j F_+(N) g_j^{-1} \rangle \right.$$
$$+ \sum_{\substack{\text{odd } i \\ i=[j\pm(1l+1)] \mod 2k \\ 0 \le l \le [d^{-2/5}/2]-1}} -Q\omega(S^3) \frac{s_j^1 s_i^2 d^8}{\operatorname{dist}(q_j, q_i)^4}$$
$$\cdot [\langle \phi_j^* g_j F_+(N) g_j^{-1}, \phi_i^* g_i F_+(N) g_i^{-1} \rangle$$
$$\left. + \langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1} \rangle] \right\}$$
$$\cdot \left\| \psi_* \left(\frac{\partial}{\partial s_j} \right) \right\|_{A(y)} \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial s_j} \right)$$

$$+\sum_{\text{odd } i}\sum_{\alpha=1}^{3}\left\{-\frac{\omega(S^3)}{2}s_i^2 d^4\langle P_-F(m,n)(q_i), [T_i^\alpha, \phi_i^* g_i F_-(N) g_i^{-1}]\rangle\right.$$

$$+\sum_{\substack{\text{even } i\\ j=[i\pm(2l+1)] \bmod 2k\\ 0\le l\le [d^{-2/5}/2]-1}} -Q\frac{\omega(S^3)}{2}\frac{s_i^2 s_j^2 d^8}{\operatorname{dist}(q_i, q_j)^4}$$

$$\cdot[\langle[T_i^\alpha, \phi_i^* g_i F_-(N) g_i^{-1}], \phi_j^* g_j F_-(N) g_j^{-1}\rangle$$

$$\left.+\langle\phi_j^* g_j F_+(N) g_j^{-1}, [T_i^\alpha, \phi_i^\alpha, \phi_i^* g_i F_+(N) g_i^{-1}]\rangle]\right\}$$

$$\cdot\|\psi_*(T_k^\alpha)\|_{A(y)}\pi_y\circ J\circ\overline{\psi}_*(T_i^\alpha)$$

$$+\sum_{\text{even } j}\sum_{\alpha=1}^{3}\left\{-\frac{\omega(S^3)}{2}\langle P_+F(m,n)(q_j), [T_j^\alpha, \phi_j^* F_+(N)(g_j^{-1}]\rangle\right.$$

$$+\sum_{\substack{\text{odd } i\\ i=[j\pm(2l+1)) \bmod 2k\\ 0\le l\le [d^{-2/5}/2]-1}} -Q\frac{\omega(S^3)}{2}\frac{s_j^2 s_i^2 d^8}{\operatorname{dist}(q_j, q_i)^4}$$

$$\cdot[\langle[T_j^\alpha, \phi_j^* g_j F_+(N) g_j^{-1}], \phi_i^* g_i F_+(N) g_i^{-1}\rangle$$

$$\left.+\langle\phi_i^* g_i F_-(N) g_i^{-1}, [T_j^\alpha, \phi_j^* g_j F_-(N) g_j^{-1}]\rangle]\right\}$$

$$\cdot\|\psi_*(T_j^\alpha)\|_{A(y)}\pi_y\circ J\circ\overline{\psi}_*(T_j^\alpha)$$

$$+\sum_{\text{odd } i}\sum_{t=1}^{4}\left\{\sum_{\substack{\text{even } j\\ j=[i\pm(2l+1)] \bmod 2k\\ 0\le l\le [d^{-2/5}/2]-1}} -Q\frac{\omega(S^3)}{2}s_i^2 s_j^2 d^8\right.$$

$$\cdot[\langle\phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1}\rangle$$

$$+\langle\phi_j^* g_j F_+(N) g_j^{-1}, \phi_i^* g_i F_+(N) g_i^{-1}\rangle]$$

$$\left.\cdot\frac{\partial}{\partial x_i^t}\operatorname{dist}(q_i, q_j)^{-4}\right\}$$

$$\cdot \left\| \psi_* \left(\frac{\partial}{\partial x_i^t} \right) \right\|_{A(y)} \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial x_i^t} \right)$$

$$+ \sum_{\text{even } j} \sum_{t=1}^{4} \left\{ \sum_{\substack{\text{odd } i \\ i=[j\pm(2l+1)] \bmod 2k \\ 0\le l \le [d^{-2/5}/2]-1}} -Q \frac{\omega(S^3)}{s}^2_j s_j^2 s_i^2 d^4 \right.$$

$$\cdot [\langle \phi_j^* g_j F_+(N) g_j^{-1}, \phi_i^* g_i F_+(N) g_i^{-1} \rangle$$

$$+ \langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1} \rangle]$$

$$\left. \cdot \frac{\partial}{\partial x_j^t} \operatorname{dist}(q_j, q_i)^{-4} \right\}$$

$$\cdot \left\| \psi_* \left(\frac{\partial}{\partial x_j^t} \right) \right\|_{A(y)} \pi_y \circ J \circ \overline{\psi}_* \left(\frac{\partial}{\partial x_j^t} \right).$$

In the above expression, the definitions of $\omega(S^3)$, ϕ_i, $F(m, n)$, and $F_\pm$ are given in §3. Set

$$f^2(y) = f(y) - f^1(y). \tag{5.5}$$

Hence $f(y) = f^1(y) + f^2(y)$. The utility of this splitting of f is in part due to the following proposition, which arises from the estimates for $\mathscr{U}_{\xi_0}(y)$ in §3. Its proof is omitted.

Proposition 5.1. *Let $f^2 \colon N_2 \to \Omega(y)$ be as in the previous definition. Then for any $y = ((s_i, f_i, g_i)_{i=1}^{2k}) \in N_2$*

$$\|f^2(y)\|_{A(y)} \le C \sum_{i=1}^{2k} \{s_i^2 d^{26/5}(C_1 + C_2 |\ln d|) + \text{higher order terms}\}. \tag{5.6}$$

Furthermore, f^1 and f^2 are U(1)*-equivariant.*

In order to utilize the decomposition of f into $f^1 + f^2$, the next proposition is necessary; it relates the vanishing of f^1 to the vanishing of f.

Proposition 5.2 (6.1 in §6 of [19]). *Let $l \in (1, 2, \cdots)$ and $n \in (0, 1, 2, \cdots)$. Let v be a C^2 map of the ball of radius $\delta > 0$, $B_\delta \subset \mathscr{R}^{n+l}$, into R^l with the following properties*:

(1) $v(0) = 0$.
(2) *$H = dv|_0$ is surjective.*
(3) *Let $\mu = |HH^*|^{1/2}$. Then $|v(x) - H(x)| < \mu \cdot \delta/2$ if $x \in B_\delta$.*

Let $v': B_\delta \to R^l$ *be continuous with* $|v'| < \mu \cdot \delta/2$. *Then there exists* $x \in B_\delta$ *such that* $v(x) + v'(x) = 0$.

Proof. Proposition 5.2 is a standard fixed point theorem. q.e.d.

We now study f^1. As in (B.10)–(B.13) of Appendix B, set

$$d\bar{x} \wedge dx = 2\sqrt{2}\sum_{\alpha=1}^{3} \omega^\alpha T^\alpha, \qquad dx \wedge d\bar{x} = -2\sqrt{2}\sum_{\alpha=1}^{3} \varpi^\alpha T^\alpha. \tag{5.7}$$

Here $\{T^\alpha\}_{\alpha=1}^3$ is an orthonormal basis of the Lie algebra of SU(2) such that

$$(T^\alpha)^2 = -1, \qquad 1 \le \alpha \le 3, \qquad T^3 = T^1 T^2. \tag{5.8}$$

In (5.7) we defined $\{\omega^\alpha, \varpi^\alpha\}$ to be

$$\begin{aligned} \omega^1 &= \sqrt{2}(dx^1 \wedge dx^2 - dx^3 \wedge dx^4)/2, \\ \omega^2 &= \sqrt{2}(dx^1 \wedge dx^3 - dx^4 \wedge dx^2)/2, \\ \omega^3 &= \sqrt{2}(dx^1 \wedge dx^4 - dx^2 \wedge dx^3)/2, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \varpi^1 &= -\sqrt{2}(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)/2, \\ \varpi^2 &= -\sqrt{2}(dx^1 \wedge dx^3 + dx^4 \wedge dx^2)/2, \\ \varpi^3 &= -\sqrt{2}(dx^1 \wedge dx^4 + dx^2 \wedge dx^3)/2. \end{aligned} \tag{5.10}$$

The $\{\varpi^\alpha\}_{\alpha=1}^3$ and $\{\omega^\alpha\}_{\alpha=1}^3$ are viewed as orthonormal bases of $\bigwedge_\pm^2 T^*\mathscr{R}^4$.

Suppose that at $\{\tilde{g}_i\}_{1\le i\le 2k}$ the expressions below take the critical values (maximum)

$$-\langle P_-F(m, n), \phi_i^* \tilde{g}_i F_-(N)\tilde{g}_i^{-1}\rangle, \qquad i = \text{odd}, \tag{5.11}$$

and

$$-\langle P_+F(m, n), \phi_j^* \tilde{g}_j F_+(N)\tilde{g}_j^{-1}\rangle, \qquad j = \text{even}. \tag{5.12}$$

Fix $f^1 = \tilde{f}_i$, $1 \le i \le 2k$, such that the ϕ_i correspond to the coordinate system $\{s, y^\alpha\}_{\alpha=1}^3$ on V_0. Since $A(m, n)$ is a reducible SU(2)-connection over $S^2 \times S^2$, one can maximize (5.11) and (5.12) with $\tilde{g}_i \equiv \tilde{g}_j \equiv \tilde{g}$; $\tilde{g}$ is independent of the positions of $\{q_i\}_{i\le i\le 2k}$. Set

$$\begin{aligned} Q_{\text{odd}} &= -\langle P_-F(m, n), \phi_i^* \tilde{g} F_-(N)\tilde{g}^{-1}\rangle, \\ Q_{\text{even}} &= -\langle P_+F(m, n), \phi_j^* \tilde{g} F_+(N)\tilde{g}^{-1}\rangle, \end{aligned} \tag{5.13}$$

where Q_{odd} and Q_{even} are positive constants. Choose $\{\tilde{q}_i\}_{i=1}^{2k}$ in the geodesic C and $\operatorname{dist}(\tilde{q}_i, \tilde{q}_j) = \pi/k = d$, $1 \le i \le 2k$. Set

(5.14)

$$\tilde{s}^2_{\text{odd}\,i} \equiv \tilde{s}^2_{\text{odd}}$$

$$= Q_{\text{even}} \left\{ Q\langle \tilde{g}F_+(N)\tilde{g}^{-1}, \tilde{g}F_+(N)\tilde{g}^{-1}\rangle \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{2}{(2l+1)^4} \right\}^{-1}$$

$$= Q_{\text{even}} \left\{ Q \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \right\}^{-1},$$

$$\tilde{s}^2_{j=\text{even}} \equiv \tilde{s}^2_{\text{even}} = Q_{\text{odd}} \left\{ Q \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \right\}^{-1}, \tag{5.15}$$

where $\tilde{s}_{\text{odd}}$ and $\tilde{s}_{\text{even}}$ are positive constants.

Set $\tilde{y} = ((\tilde{s}_i, \tilde{f}_i, \tilde{g}_i)_{i=1}^{2k}) \in \overline{N}_2$. Then $\tilde{s}_{i=\text{odd}} = \tilde{s}_{\text{odd}}$ and $\tilde{s}_{j=\text{even}} = \tilde{s}_{\text{even}}$. It is not hard to see $H_1(y)$ at $\tilde{y} \in \widetilde{N}_2$ takes the critical value $H_1(\tilde{y}) > 0$ when k is large; so $f^1(\tilde{y}) = 0$. This uses the fact that the points $\{\tilde{q}_i, \tilde{q}_{i+1}\}$ lie on a geodesic.

We now need to study the derivative of f^1 at $y = \tilde{y}$. By direct calculation, we have

(5.16)

$$\frac{\partial}{\partial s_i} T_j^\alpha H_1(y)|_{y=\tilde{y}} = \frac{\partial}{\partial s_i} \cdot \frac{\partial}{\partial x_j^t} H_1(y)|_{y=\tilde{y}} = \frac{\partial}{\partial x_i^t} T_i^\alpha H_1(y)|_{y=\tilde{y}} = 0,$$

$$1 \le i, j \le 2k, \quad 1 \le \alpha \le 3, \quad 1 \le t \le 4;$$

$$T_i^{\alpha_1} T_j^{\alpha_2} H_1(y)|_{y=\tilde{y}} = \frac{\partial^2}{\partial x_i^{t_1} \partial x_j^{t_2}} H_1(y)|_{y=\tilde{y}} = 0,$$

$$1 \le i, j \le 2k, \quad \alpha_1 \ne \alpha_2, \quad t_1 \ne t_2;$$

$$\frac{\partial^2}{\partial s_i \partial s_j} H_1(y)|_{y=\tilde{y}} = -192\omega(S^3) Q \tilde{s}_{\text{odd}} \tilde{s}_{\text{even}} d^4 \frac{1}{|(i-j) \bmod 2k|^4} \tag{5.17}$$

if $i = \text{odd}$, $j = \text{even}$, and $|(i-j) \bmod 2k| \le 2[\frac{1}{2} d^{-2/5}] - 1$;

$$\frac{\partial^2}{\partial s_i \partial s_j} H_1(y)|_{y=\tilde{y}} = 0 \quad \text{for any other case;}$$

(5.18)

$$T_i^1 T_k^1 H_1(y)|_{y=\tilde{y}} = 256\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^4 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \quad \text{for } 1 \le i \le 2k;$$

$$T_i^1 T_j^1 H_1(y)|_{y=\tilde{y}} = -128\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^4 \frac{1}{|(i-j) \bmod 2k|^4}$$

$$\text{if } i = \text{odd},\ j = \text{even},\ \text{and } |(i-j) \bmod 2k| \le 2[\tfrac{1}{2} d^{-2/5}] - 1;$$

$$T_k^1 (T_j^1 H_1(y)|_{y=\tilde{y}} = 0 \quad \text{for any other case};$$

(5.19)

$$T_i^\alpha T_i^\alpha H_1(y)|_{y=\tilde{y}} = -256\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^4 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \quad \text{for } \alpha = 2, 3,\ 1 \le i \le 2k;$$

$$T_i^\alpha T_j^\alpha H_1(y)|_{y=\tilde{y}} = -128\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^4 \frac{1}{|(i-j) \bmod 2k|^4}$$

$$\text{if } \alpha = 2, 3,\ i = \text{odd},\ j = \text{even},\ \text{and } |(i-j) \bmod 2k| \le 2[\tfrac{1}{2} d^{-2/5}] - 1;$$

$$T_k^\alpha T_j^\alpha H_1(y)|_{y=\tilde{y}} = 0 \quad \text{for any other case};$$

(5.20)

$$\frac{\partial^2}{\partial x_i^1 \partial x_i^1} H_1(y)|_{y=\tilde{y}} = -1920\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \quad \text{for } 1 \le i \le 2k;$$

$$\frac{\partial^2}{\partial x_i^1 \partial x_j^1} H_1(y)|_{y=\tilde{y}} = 960\omega(S^3)Q\tilde{s}_{\mathrm{odd}}^2\tilde{s}_{\mathrm{even}}^2\, d^2 \frac{1}{|(i-j) \bmod 2k|^4}$$

$$\text{if } i = \text{odd},\ j = \text{even},\ \text{and } |(i-j) \bmod 2k| \le 2[\tfrac{1}{2} d^{-2/5}] - 1;$$

$$\frac{\partial^2}{\partial x_i^1 \partial x_j^1} H_1(y)|_{y=\tilde{y}} = 0 \quad \text{for any other case};$$

(5.21)

$$\frac{\partial^2}{\partial x_i^t \partial x_i^t} H_1(y)|_{y=\tilde{y}} = 384\omega(S^3) Q \tilde{s}_{\text{odd}}^2 \tilde{s}_{\text{even}}^2 d^2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4}$$
for $2 \le t \le 4$, $1 \le i \le 2k$;

$$\frac{\partial^2}{\partial x_i^t \partial x_j^t} H_1(y)|_{y=\tilde{y}} = -192\omega(S^3) Q \tilde{s}_{\text{odd}}^2 \tilde{s}_{\text{even}}^2 d^2 \frac{1}{|(i-j) \bmod 2k|^4}$$
if $2 \le t \le 4$, $i = \text{odd}$, $j = \text{even}$, and $|(i-j) \bmod 2k| \le 2[\frac{1}{2} d^{-2/5}] - 1$;

$$\frac{\partial^2}{\partial x_i^t \partial x_j^t} H_1(y)|_{y=\tilde{y}} = 0, \quad \text{for any other case.}$$

Proposition 4.6 and equalities (5.14)–(5.21) give the proposition below.

Proposition 5.3. *Fix parameters* $f_i = \tilde{f}_i$ *and let* $\tilde{y} = ((\tilde{s}_i, \tilde{f}_i, \tilde{g}_i)_{i=1}^{2k}) \in N_2$. *Here*

($\dot{1}$) $\tilde{q}_i$ *is in the geodesic* C, $1 \le i \le 2k$, *and* $\operatorname{dist}(\tilde{q}_{i+1}, \tilde{q}_i) = \pi/k$;

($\dot{2}$) $\tilde{g}_i = \tilde{g}$, $1 \le i \le 2k$, *such that the expressions below take the critical values* (*maximum*):

$$Q_{\text{odd}} = -\langle P_-(m, n), \phi_i \tilde{g}_i F_-(N) \tilde{g}_k^{-1} \rangle, \quad i = \text{odd},$$

$$Q_{\text{even}} = -\langle P_+(m, n), \phi_j \tilde{g}_j F_+(N) \tilde{g}_j^{-1} \rangle, \quad j = \text{even};$$

($\dot{3}$)
$$\tilde{s}_{i=\text{odd}}^2 \equiv \tilde{s}_{\text{odd}}^2 = Q_{\text{even}} \left\{ 192 Q \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \right\}^{-1},$$

$$\tilde{s}_{j=\text{even}}^2 \equiv \tilde{s}_{\text{even}}^2 = Q_{\text{odd}} \left\{ 192 Q \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} \right\}^{-1}.$$

Then $H_1(y)$ *at* $y = \tilde{y}$ *takes a critical value and* $f^1(\tilde{y}) = 0$. *We denote the tangent map of* f^1 *at* $\tilde{y}$ *by* $H = \nabla f^1|_{y=\tilde{y}}$. *Then there exists the following* *expression:*

(5.22)
$$\begin{aligned} H = & -192C_s Q\omega(S^3)\tilde{s}^2_{\text{odd}}\tilde{s}^2_{\text{even}}\, d^4 E_{2k\times 2k} \\ & \oplus 128 C_{T^1} Q\omega(S^3)\tilde{s}^2_{\text{odd}}\tilde{s}^2_{\text{even}}\, d^4 \\ & \quad \cdot \left(2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} I_{2k\times 2k} - E_{2k\times 2k}\right) \\ & \oplus \sum_{\alpha=2}^{3} 128 C_{T^\alpha} Q\omega(S^3)\tilde{s}^2_{\text{odd}}\tilde{s}^2_{\text{even}}\, d^4 \\ & \quad \cdot \left(-2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} I_{2k\times 2k} - E_{2k\times 2k}\right) \\ & \oplus 960 C_{x^1} Q\omega(S^3)\tilde{s}^2_{\text{odd}}\tilde{s}^2_{\text{even}}\, d^4 \\ & \quad \cdot \left(-2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} I_{2k\times 2k} + E_{2k\times 2k}\right) \\ & \oplus \sum_{t=2}^{3} 192 C_{x^t} Q\omega(S^3)\tilde{s}^2_{\text{odd}}\tilde{s}^2_{\text{even}}\, d^4 \\ & \quad \cdot \left(2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4} I_{2k\times 2k} - E_{2k\times 2k}\right), \end{aligned}$$

where C_s, C_{T^α}, *and* C_{x^t} *are positive constants,* $I_{2k\times 2k}$ *is the identity, and* $E_{2k\times 2k}$ *is*

(5.23)
$$E_{ij} = \begin{cases} \dfrac{1}{|(i-j) \bmod 2k|^4} & \textit{if } i = \text{odd},\ j = \text{even},\ \textit{and} \\ & |(i-j) \bmod 2k| \le 2[\tfrac{1}{2} d^{-2/5}] - 1, \\ 0 & \textit{for any other case.} \end{cases}$$

A priori estimates for eigenvalues of $E_{2k\times 2k}$ are given as follows.

Proposition 5.4. *Let* $E_{2k\times 2k}$ *be the* $2k \times 2k$ *matrix which is defined by* (5.23). *Then for* $E_{2k\times 2k}$, *the following properties hold*:

(1) *When* $k = \text{even}$, $\det E_{2k\times 2k} = 0$.

(2) *When* $k = \text{odd}$, *the eigenvalues can be written as*

(5.24)
$$\lambda_i = 2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{\cos(i-1)(2l+1)\pi/k}{(2l+1)^4} \quad \textit{for } 1 \le i \le 2k,$$

$$(5.25) \qquad \lambda_{\max} = 2 \sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4},$$

$$(5.26) \qquad |\lambda_i| \geq \frac{3}{2}\left|\cos(i-1)\frac{\pi}{k}\right| \quad \textit{for } 1 \leq i \leq 2k.$$

The proof of the above proposition is given in Appendix C.

We are now able to prove Theorems 1.1 and 1.2 using Propositions 5.1–5.4:

Proof of Theorem 1.1. Note that $f = f^1 + f^2$ is a U(1)-equivariant map from TN_2 to $\Omega(y)$. Suppose that $\tilde{y} = ((\tilde{s}_i, \tilde{f}_i, \tilde{g}_i)_{i=1}^{2k}) \in \overline{N}_2$. Then $f^1(\tilde{y}) = 0$. According to Proposition 5.4, when k = odd the tangent map $H = df^1|_{y=\tilde{y}}$ of f^1 contains five null eigenvectors $V_{T^1} = \{T_1^1, T_2^1, \cdots, T_{2k}^1\}$ and $v_{x^\beta} = \{\partial/\partial x_1^\beta, \partial/\partial x_2^\beta, \cdots, \partial/\partial x_{2k}^\beta\}$, $1 \leq \beta \leq 4$. Since $f = f^1 + f^2$ is a U(1)-equivariant map, $f|_{y=\tilde{y}}$ restricted to V_{T^1} takes zero. Now recall the construction of the approximate solution space: The geodesic C is the largest circle on the first factor, and $\{s, y^\alpha\}_{\alpha=1}^3$ is the coordinate system on the neighborhood V_0 of C, $\{\psi_*(\partial/\partial s), \psi_*(\partial/\partial y^\alpha)\}_{\alpha=1}^3 = \{V_{x^\beta}\}_{\beta=1}^4$. Denote by $T_y^\perp \overline{N}_2$ the complement of V_{T^1} and V_{x^β}, $1 \leq \beta \leq 4$. Then $H|_{T_y^\perp \overline{N}_2}$ is nondegenerate and $\|H|_{T_y^\perp \overline{N}_2}\|_{A(y)} = Cd^5$. On the other hand, f^2 obeys $\|f^2\|_{A(y)} = \sum_{i=1}^{2k} s_i^2 d^{26/5}(C_1 + C_2|\ln d|)$. Hence, if k = odd and k is sufficiently large (i.e., d is sufficiently small), then f^1 and f^2 on $T_y^\perp \overline{N}_2$ satisfy the conditions in Proposition 5.2. So, there exists $\tilde{y}' \in \overline{N}_2$ nearby to $\tilde{y}$ such that $f(\tilde{y}')|_{T_{y'}^\perp \overline{N}_2} = 0$. In fact, due to the symmetries of $S^2 \times S^2$, $\tilde{y}' = ((\tilde{s}_i', \tilde{f}_i', \tilde{g}_i')_{i=1}^{2k})$ has the following properties in the coordinate system $\{s, y^\alpha\}_{\alpha=1}^3$:

(1) $S(\pi(\tilde{f}_{i+1}')) - S(\pi(\tilde{f}_i')) = S(\pi(\tilde{f}_i')) - S(\pi(\tilde{f}_{i-1}'))$;

(2) $y^\alpha(\pi(\tilde{f}_{i+1}')) = y^\alpha(\pi(\tilde{f}_i'))$, $1 \leq \alpha \leq 3$.

On the other hand, $\{y^2, y^3\}$ are the parameters of the geodesic on $S^2 \times S^2$. Therefore $\mathrm{YM}(A(\tilde{y}' + \mathscr{U}_{\xi_0}(\tilde{y}')))$ is an even function of y^1, and is independent of the parameters $\{y^2, y^3\}$. Thus, $A(\tilde{y}) + \mathscr{U}_{\xi_0}(\tilde{y})$ is exactly a solution to Yang-Mills equations on $S^2 \times S^2$.

Recall that we have already established that it is an irreducible SU(2)-connection on the principal SU(2)-bundle over $S^2 \times S^2$ with degree

(5.27) $$-\int_{S^2\times S^2} C_2(P) = 2mn\,.$$

By inspection, $A(\tilde{y}) + \mathscr{U}_{\xi_0}(\tilde{y})$ is neither a self-dual nor anti-self-dual connection, nor invariant under any nondiscrete group of isometries. Furthermore, $A(\tilde{y}) + \mathscr{U}_{\xi_0}(\tilde{y})$ is not a local minimal solution; this is guaranteed by the next proposition.

Proposition 5.5 (Theorem B′ in [6]). *Any weakly stable Yang-Mills field with group* SU(2) *on any compact orientable homogeneous Riemannian* 4-*manifold is either self-dual, or anti-self-dual, or reduced to an abelian field.*

Remark 5.1. In fact, the index of $\nabla^2\,\mathrm{YM}$ at $A(\tilde{y})+\mathscr{U}_{\xi_0}(\tilde{y})$ is equal to the index of $\nabla^2\,\mathrm{YM}$ at $A(m\,,\,n)$.

Proof of Theorem 1.2. For $S^1\times S^3$, the argument is almost the same as for $S^2\times S^2$. We need remark that we have chosen the S^1 of $S^1\times S^3$ as our simple, closed geodesic. Since the background connection is an irreducible connection, $N_2\to\mathscr{B}^\sharp(P)$ is an embedding. Hence, $H = df^1|_{\tilde{y}}$ has only four null eigenvectors $\{V_{x^\beta}\}_{\beta=1}^4$; all correspond to isometries of $S^1\times S^3$.

Appendix A. The isolated nonminimal SU(2)-connections

In this appendix we shall construct reducible nonminimal SU(2)-connections over $S^2\times S^2$ and irreducible nonminimal SU(2)-connections over $S^1\times S^3$ which are isolated, and list their properties. For more details, refer to [24].

It is well known that S^2 is diffeomorphic to the complex projective space $\mathbf{CP}^1\cong C^2\backslash\{o\}/C^*$ viewed as the set of 1-dimensional linear subspaces in C^2. There exists a tautological line bundle L over S^2 whose first Chern number is

(A.1) $$\int_{\mathbf{CP}^1} C_1(L) = -1\,,$$

where $C_1(L)$ is the first Chern class of L. Consider the standard metric on $S^2\cong\mathbf{CP}^1$; then the first Chern class is written as

(A.2) $$C_1(L) = -\frac{1}{4\pi}\omega\,,$$

where ω is the volume form on $\mathbf{CP}^1$.

Suppose that A_0 is the canonical connection on L; then the curvature of A_0 is

$$F_{A_0} = i\omega/2. \tag{A.3}$$

Set $L(m, n) = \pi_1^* L^m \otimes \pi_2^* L^n \to \mathbf{CP}^1 \times \mathbf{CP}^1$, which is a linear bundle over the product manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$, where $m, n \in \mathscr{Z}$, and π_1 and π_2 are the projective operators from the product space $\mathbf{CP}^1 \times \mathbf{CP}^1$ to the first factor and the second factor respectively. We have the following diagram:

$$\begin{array}{ccc} & \mathbf{CP}^1 \times \mathbf{CP}^1 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbf{CP}^1 & & \mathbf{CP}^1 \end{array} \tag{A.4}$$

It is clear that the first Chern class of the line bundle $L(m, n)$ is

$$C_1(L(m, n)) = -\tfrac{1}{4}(m\omega_1 + n\omega_2), \tag{A.5}$$

where $\omega_1 = \pi_1^*\omega$ and $\omega_2 = \pi_2^*\omega$

Let $A(m, n) = \pi_1^*(\otimes^m A_0) \otimes \pi_2^*(\otimes^n A_0)$. Then the corresponding curvature is

$$F(m, n) = F_{A(m,n)} = \frac{i}{2}(m\omega_1 + n\omega_2). \tag{A.6}$$

Hence $L(m, n) \oplus L(m, n)^{-1} \to \mathbf{CP}^1 \times \mathbf{CP}^1$ is a reducible SU(2)-bundle over $\mathbf{CP}^1 \times \mathbf{CP}^1$, and there exists a reducible SU(2)-connection whose curvature is

$$F(m, n) = \frac{1}{2}(m\omega_1 + n\omega_2)\begin{pmatrix} i & o \\ o & -i \end{pmatrix}. \tag{A.7}$$

Since

$$C_1(L(m, n) \oplus L(m, n)^{-1}) = 0$$

and

$$-C_2(L(m, n) \oplus L(m, n)^{-1}) = \frac{1}{4\pi^2}\det(F(m, n)) = \frac{mn}{8\pi^2}\omega_1 \wedge \omega_2,$$

if $m = \pm n$ then $A(m, n)$ is a reducible (anti)-self-dual SU(2)-connection on $L(m, n) \oplus L(m, n)^{-1}$ with instanton number $\pm 2m^2$. When $|m| \neq |n|$, then $A(m, n)$ is a reducible nonminimal Yang-Mills connection on $L(m, n) \oplus L(m, n)^{-1}$ with degree $2mn$ since $F(m, n)$ is neither self-dual nor anti-self-dual. For details on the line bundle over the complex projective space, readers are referred to [11].

For simplicity, set $\eta(m, n) = L(m, n) \oplus L(m, n)^{-1}$. Then

$$\mathrm{Ad}\,\eta(mn) = i\mathscr{R} \oplus L(m, n)^2 .$$

The second variation $\nabla^2 \mathrm{YM}_{A(m,n)}(\cdot, \cdot)$ of the Yang-Mills functional YM at $A(m, n)$ is

$$\mathrm{YM}_{A(m,n)}(a, b) = \int_{\mathbf{CP}^1 \times \mathbf{CP}^1} (D_A a, D_A b) + (F_A, [a, b]) \tag{A.8}$$

for $a, b \in \Gamma(\mathrm{Ad}\,\eta(m, n) \otimes T^*\mathbf{CP}^1 \times \mathbf{CP}^1)$. Because the Yang-Mills equations are $\mathrm{Aut}\,\eta(m, n)$ invariant, the Hessian has an infinite-dimensional null space: If A is a solution to the Yang-Mills equations, then $\nabla^2 \mathrm{YM}_A(a, \cdot) = 0$ for all $a = D_A\phi$ with $\phi \in \Gamma(\mathrm{Ad}\,\eta(m, n))$. To obtain elliptic equations, such an approach is used here as in [2]. For the reducible connection $A(m, n)$, consider the bilinear form

$$\begin{aligned} &\overline{\nabla}^2 \mathrm{YM}_{A(m,n)}(a, b) \\ &\quad = \nabla^2 \mathrm{YM}_{A(m,n)}(a, b) + \langle D_{A(m,n)} * a, D_{A(m,n)} * b \rangle_2 . \end{aligned} \tag{A.9}$$

We have the corresponding elliptic operator

$$D_A^* D_A a + D_A D^* A a + *[*F_A, a], \tag{A.10}$$

where $A = A(m, n)$ is a reducible connection on $L(m, n)$. Note that if $a \in \Gamma(i\mathscr{R} \otimes T^*S^2 \times S^2)$, $*[*F_A, a] = 0$, while $a \in \Gamma(L(m, n) \otimes T^*S^2 \times S^2)$, then $*[*F_A, a] - 2 * (*F_A \wedge a)$, where $F_A = \frac{i}{2}(m\omega_1 + n\omega_2)$. In order to consider the isolated phenomenon of $A(m, n)$, we must study the spectrum of the elliptic operator

$$D_A^* D_A a + D_A D_A^* a + 2 * (*F_A \wedge a) .$$

Since $T^*S^2 \times S^2 = \pi_1^* T^*S^2 \oplus \pi_2^* T^*S^2$, we can compute the spectrum of the above elliptic operator by using the method of separation of variables. For $a \in \Gamma(L(m, n)^2 \otimes \pi_1^* T^*S^2)$ set $a = a_1 \otimes b_1$, where $a_1 \in \pi_1^*\Gamma(L^{2m} \otimes T^*S^2)$ and $b_1 \in \pi_2^*\Gamma(L^{2n})$. By direct calculation, we have

$$\begin{aligned} &D_A^* D_A a + D_A D_A^* a + 2 * (*F_A \wedge a) \\ &\quad = (D_A^* D_A a_1 + D_A D_A^* a_1) \otimes b_1 + a_1 \otimes (D_A^* D_A b_1) \\ &\qquad + i * [(m\omega_2 + n\omega_1) \wedge a_1 \otimes b_1] . \end{aligned}$$

So the operator (A.10) splits into two operators

$$D_A^* D_A a_1 + D_A D_A^* a_1 + mi * a_1 \tag{A.11}$$

and

(A.12) $$D_A^* D_A b_1 .$$

Similarly, when $a \in \Gamma(L(m, n)^2 \otimes \pi_2^* T^* S^2)$, set $a = a_2 \otimes b_2$, where $a_2 \in \pi_1^* \Gamma(L^{2m})$ and $b_2 \in \pi_2^* \Gamma(L^{2n} \otimes T^* S^2)$. For this we have the following two operators:

(A.13) $$D_A^* D_A a_2$$

and

(A.14) $$D_A^* D_A b_2 + D_A D_A^* b_2 + ni * b_2 .$$

Hence, the main problem is to investigate the spectrum of the operators

(A.15) $$D_A^* D_A a = \lambda a \quad \text{for } a \in \Gamma(L^{2n})$$

and

(A.16) $$D_A^* D_A b + D_A D_A^* b + ni * b = \lambda b \quad \text{for } b \in \Gamma(L^{2n} \otimes T^* S^2) .$$

We now consider the elliptic operators on the Riemannian surfaces. it is well known that when the base manifold of the complex vector bundle is two-dimensional, the Laplace equation naturally relates holomorphic structures and can therefore be understood best in holomorphic context.

To see this, recall that when $\dim M = 2$, the $*$ operator of a Riemannian surface on M maps Ω^1 to Ω^1, with $*^2 = -1$. Hence we have a natural decomposition

$$\Omega_c^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

with Ω_c complex, $\Omega \otimes C$, and

$$* = -i \quad \text{on } \Omega^{1,0}, \qquad * = i \quad \text{on } \Omega^{0,1}$$

of the complexified de Rham complex. This decomposition splits $d\colon \Omega^0 \to \Omega^1$ into $d'\colon \Omega^0 \to \Omega^{1,0}$ and $d''\colon \Omega^0 \to \Omega^{0,1}$ and so induces a holomorphic structure on M; a holomorphic function f corresponds (locally) to solutions of $d'' f = 0$.

Suppose now that V is a complex vector bundle over M, and A is a connection for V. Then the above argument can be applied to the complex $\Omega^*(M, V)$ and D_A, giving a decomposition

$$\Omega_c^1(M, V) = \Omega^{1,0}(M, V) \oplus \Omega^{0,1}(M, V)$$

according to eigenvalues of $*$. There is a corresponding decomposition of D_A, so that we have the diagram

$$\text{(A.17)}\qquad \begin{array}{ccc} \Omega^{1,0}(M,V) & \xrightarrow{D''_A} & \Omega^2_c(M,V) \\ \uparrow{\scriptstyle D'_A} & & \uparrow{\scriptstyle D'_A} \\ \Omega^0_c(M,V) & \xrightarrow{D''_A} & \Omega^{0,1}(M,V) \end{array}$$

which is of course compatible with the corresponding decomposition of $\Omega_c(M)$, and now the operator D''_A defines a holomorphic structure on the vector bundle V over M. This can be proved as in [3, Theorem 5.1] by applying the Newland-Nirenberg integrability theorem for complex structure.

In order to study the spectrum of $\Delta_A = D^*_A D_A + D_A D^*_A$, we have to use our decomposition of Ω^*_c and the corresponding decomposition of D_A into $D'_A + D''_A$. We want to compute the spectrum of Δ_A in terms of the dimensions of the harmonic forms in $\Omega^{1,0}$ and $\Omega^{0,1}$. Now in the diagram (A.17) each arrow has a natural adjoint, and we can therefore associate a Laplacian with each arrow. Each such Laplacian gives a self-adjoint operator on the spaces at both ends of the arrow. Thus we have a lower and upper $\square''_A$ defined by

$$\text{(A.18)}\qquad \square''_A = D''_A(D''_A)^* + (D''_A)^* D''_A,$$

as well as left and right $\square'_A$ defined by

$$\text{(A.19)}\qquad \square'_A = D'_A(D'_A)^* + (D'_A)^* D'_A.$$

Now the basic relation between these operators is given by the following.

Lemma A.1 [2, Lemma 5.9]. *The Laplacians* $\square'_A$ *and* $\square''_A$ *induce the same operator on* $\Omega^{1,0}$ *and* $\Omega^{0,1}$. *Further,* $\Delta_A = D_A D^*_A + D^*_A D_A$ *preserves these and*

(i) $\Delta_A = 2\square'_A = 2\square''_A$ *on* $\Omega^{1,0}$ *and* $\Omega^{0,1}$ *while*

(ii) $\Delta_A = \square'_A + \square''_A$ *on* $\Omega^{0,0}$ *and* $\Omega^{1,1}$ *and, finally, on* $\Omega^{0,0}$ *these two Laplacians differ by* $i * F_A$:

(iii) $\square'_A - \square''_A = i * F_A$.

In the special case that $M = S^2$, $V = L^{2n}$, *we have*

$$\square'_A - \square''_A = -n \quad \text{on } \Gamma(L^{2n}).$$

In terms of Lemma A.1, for the operators (A.15) and (A.16) the following equivalent relations hold:

(A.20) $$D_A^* D_A = 2\Box_A'' - n \quad \text{on } \Gamma(L^{2n}),$$

(A.21) $$D_A^* D_A + D_A D_A^* + ni* = 2\Box_A'' + n \quad \text{on } \Omega^{1,0}(L^{2n}),$$

(A.22) $$D_A^* D_A + D_A D_A^* + ni* = 2\Box_A'' - n \quad \text{on } \Omega^{0.1}(L^{2n}).$$

In order to study the spectrum of $\Box_A''$, we require.

Lemma A.2. *Suppose that* $\lambda \geq 0$. *Let* $E_\lambda(L^{2n}; p, q)$ *be the eigenform subspace of* $\Box_A''$ *on the* (p, q)*-form with values* L^{2n}, *i.e.,* $\forall \alpha \in E_\lambda(L^{2n}; p, q)$, $\Box_A''\alpha = \lambda\alpha$. *Then we have*

(A.23) $$\sum_{q=0}^{1}(-1)^q \dim E_0(L^{2n}; p, q) = \chi(S^2; L^{2n}, p) = \frac{1}{2}\chi(S^2) + C_1(L^{2n} \otimes \textstyle\bigwedge^p T_h^*) = 1 - 2n - 2p,$$

(A.24) $$\sum_{q=0}^{1}(-1)^q \dim E_\lambda(L^{2n}; p, q) = 0 \quad \text{as } \lambda > 0, \; p = 0, 1.$$

Remark A.1. (A.23) is just the Riemann-Roch Theorem [11].

By a standard vanishing theorem and Kodaira-Serre duality ([11], [13]), we get

Proposition A.1. *Let* L *be the tautological line bundle over* $S^2 \cong \mathbf{CP}^1$. *Suppose* A *is the canonical connection on* L. *Then we have the following results*:

(1) *If* $n > 0$, *then*

$$\dim E_0(L^{2n}; 0, 0) = 0, \qquad \dim E_0(L^{2n}; 0, 1) = 2n - 1,$$
$$\dim E_0(L^{2n}; 1, 0) = 0, \qquad \dim E_0(L^{2n}; 1, 1) = 2n + 1.$$

(2) *If* $n < 0$, *then*

$$\dim E_0(L^{2n}; 0, 0) = -2n + 1, \qquad \dim E_0(L^{2n}; 0, 1) = 0,$$
$$\dim E_0(L^{2n}; 1, 0) = -2n - 1, \qquad \dim E_0(L^{2n}; 1, 1) = 0.$$

In the special case $n = 0$, *we can directly compute the spectrum of* $\Box_A''$.

Lemma A.3 (see [5]). *The spectrum of* $\Box_A''$ *on* S^2 *is* $\lambda_k = k(k+1)/2$ *with the multiplicity* $2k + 1$.

Remark A.2. In the above lemma, S^2 admits the standard metric. However, in [5] $\mathbf{CP}^1$ takes the Fubini-Study metric as a complex projective space whose holomorphic curvature is 4. The eigenvalue of $\square''_A$ with respect to the standard metric is slightly different from it with respect to the Fubini-Study metric.

We now return to the general case. To generalize the above lemma, consider isomorphisms

$$i*: \Omega^{1,1}((L^{2n}) \to \Omega^{0,0}(L^{2n}). \tag{A.25}$$

By $\lambda(n; p, q)$ we denote the kth eigenvalues of $\square''_A$ on $\Omega^{p,q}(L^{2n})$, $p, q = 0, 1$. In particular, $\lambda_0(n; p, q) = 0$. By $E_k(n; p, q)$ we denote the eigenspace of $\square''_A$ with eigenvalue $\lambda_k(n; p, q)$ on $\Omega^{p,q}(L^{2n})$. For any $\alpha \in E_k(n; 0, 0)$, we have

$$\begin{aligned}\square''_A(-i*\alpha) &= D''_A(D''_A)^*(-i*\alpha) = iD''_A * D'_A\alpha \\ &= -i*(D'_A)*D'_A\alpha = (\lambda_k(n; 0, 0) + i*F_A)(-i*\alpha).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda_{k-1}(n; 1, 1) &= \lambda_k(n; 0, 0) - n, \\ \dim E_{k-1}(n; 1, 1) &= \dim E_k(n; 0, 0)\end{aligned} \tag{A.26}$$

as $n > 0$, and

$$\begin{aligned}\lambda_{k+1}(n; 1, 1) &= \lambda_k(n; 0, 0) - n, \\ \dim E_{k+1}(n; 1, 1) &= \dim E_k(n; 0, 0)\end{aligned} \tag{A.27}$$

as $n < 0$. By the Bochner technique [13], we have the Weitzenbock formula on $\Omega^{1,1}(L^{2n})$:

$$\square''_A = -\nabla_{L^{2n}\otimes T^*_h}\overline{\nabla}_{L^{2n}\otimes T^*_h}, \tag{A.28}$$

where $\nabla_{L^{2n}\otimes T^*_h}$ is the covariant differential with respect to the connection on $L^{2n}\otimes T^*_h$ which is given by a tensor product of the canonical connection on L^{2n} and Riemannian connection over S^2. A much subtler theorem, proved in versions over the years by Hilbert, Birkhoff, Grothendieck, and others ([1], [12]) asserts that every holomorphic vector bundle over $\mathbf{CP}^1$ is isomorphic to a direct *sum* of L^{k_i}, and the integers $(k_1, \cdots, k_m)$ are unique up to permutation. In particular, $T^*_h S^2 \cong L^2$ since $C_1(T^*_h S^2) = -2$, where $T^*_h S^2$ is the holomorphic cotangent bundle of S^2. In our case $L^{2n} \otimes T^*_h S^2 \cong L^{2n+2}$. Note that the holomorphic curvature of the Fubini-Study metric is the constant 4, and the standard metric on S^2

and Riemannian metric induced by the Fubini-Study metric on $\mathbf{CP}^1$ only differ by a conformal constant. In fact, as a line bundle, the connection on $L^{2n} \otimes T_h^* S^2$ is equivalent to the canonical connection on L^{2n+2}. Hence $-\nabla_{L^{2n}\otimes T_h^*} \overline{\nabla}_{L^{2n}\otimes T_h^*}$ is viewed as the elliptic operator $\square_A''$ on $\Omega^{0,0}(L^{2n+2})$. In terms of (A.26), (A.27), and (A.28), according to Lemmas A.2 and A.3 and Proposition A.1, we get the following proposition by induction on n starting with $n = 0$.

Proposition A.2. *Let L be the tautological line bundle over the standard 2-sphere S^2. Suppose that A is the canonical connection on L. Then the spectrum of $\square_A''$ on $\Omega^{p,q}$, (L^{2n}) and its multiplicity are respectively as follows*:

As $n > 0$

(1) $\dim E_0(n; 0, 0) = 0$, $\dim E_0(n; 0, 1) = 2n - 1$ *for* $k \geq 1$,
(2) $\lambda_k(n; 0, 0) = \lambda_k(n; 0, 1) = \frac{1}{2}[(n+k)(n+k-1) - n(n-1)]$,
(3) $\dim E_k(n; 0, 0) = \dim E_k(n; 0, 1) = 2(n+k) - 1$,
(4) $\dim E_0(n; 1, 0) = 0$, $\dim E_0(n; 1, 1) = 2n + 1$ *for* $k \geq 1$
(5) $\lambda_k(n; 1, 0) = \lambda_k(n; 1, 1) = \frac{1}{2}[(n+k+1)(n+k) - (n+1)n]$,
(6) $\dim E_k(n; 1, 0) = \dim E_k(n; 1, 1) = 2(n+k) + 1$.

As $n < 0$

(1) $\dim E_0(n; 0, 0) = -2n + 1$, $\dim E_0(n; 0, 1) = 0$ *for* $k \geq 1$,
(2) $\lambda_k(n; 0, 0) = \lambda_k(n; 0, 1) = \frac{1}{2}[(k+1-n)(k-n) + n(1-n)]$,
(3) $\dim E_k(n; 0, 0) = \dim E_k(n; 0, 1) = 2(k-n) + 1$
(4) $\dim E_0(n; 1, 0) = -2n - 1$, $\dim E_0(n, 1, 1) = 0$ *for* $k \geq 1$
(5) $\lambda_k(n; 1, 0) = \lambda_k(n; 1, 1) = \frac{1}{2}[(k-n-1)(k-n) - n(n+1)]$,
(6) $\dim E_k(n; 1, 0) = \dim E_k(n; 1, 1) = 2(k-n) - 1$.

In this appendix our purpose is to construct a nonminimal isolated solution to the Yang-Mills equations which is reducible. By separation of variables, we need to compute the spectrum of $D_A^* D_A + D_A D_A^*$ on $\Gamma(L^{2n})$ and $\Gamma(L^{2n} \otimes T^* S^2)$. According to Lemma A.1, the issue is reduced to a computation of the spectrum of

$$D_A^* D_A + D_A D_A^* = 2\square_A'' - n, \quad \text{on } \Omega^{0,0}(L^{2n}), \tag{A.29}$$

$$D_A^* D_A + D_A D_A^* + ni* = 2\square_A'' + n, \quad \text{on } \Omega^{1,0}(L^{2n}), \tag{A.30}$$

$$D_A^* D_A + D_A D_A^* + ni* = 2\square_A'' - n, \quad \text{on } \Omega^{0,1}(L^{n2}). \tag{A.31}$$

We define $\tilde{\lambda}_0(n; p, q) < \tilde{\lambda}_1(n; p, q) < \cdots < \tilde{\lambda}_k(n; p, q) < \cdots$ as the spectrum of the operators (A.29), (A.30), and (A.31) respectively, and define $\widetilde{E}_k(n; p, q)$, $k \geq 0$, as the corresponding eigensubspace. As a corollary of Proposition A.2, we have the following proposition.

Proposition A.3. *For $n \geq 0$, $D_A^* D_A = 2\square_A'' \mp n$ on $\Omega^{0,0}(L^{\pm 2n})$ has the spectrum*

$$\tilde{\lambda}_k(\pm n\,;0,0) = n(2k+1) + k(k+1) \quad \text{for } k \geq 0 \tag{A.32}$$

with multiplicity,

$$\dim \widetilde{E}_k(\pm n\,;0,0) = 2(n+k)+1 \quad \text{for } k \geq 0. \tag{A.33}$$

For $n \geq 1$, $D_A^* D_A + D_A D_A^* \pm ni*$ on $\Omega^{1,0}(L^{\pm 2n})$ and $\Omega^{0,1}(L^{\pm 2n})$ has the spectrum

$$\tilde{\lambda}_k(n\,;0,1) = \tilde{\lambda}_k(-n\,;1,0) = n(2k-1) + (k-1) \quad \text{for } k \geq 0 \tag{A.34}$$

with multiplicity

$$\dim \widetilde{E}_k(n\,;0,1) = \dim \widetilde{E}_k(-n\,;1,0) = 2(n+k)-1 \quad \text{for } k \geq 0, \tag{A.35}$$

and the spectrum

$$\tilde{\lambda}_k(n\,;1,0) = \tilde{\lambda}_k(-n\,;-,1) = (2k+1)n + k(k+1) \quad \text{for } k \geq 0 \tag{A.36}$$

with multiplicity

$$\dim \widetilde{E}_k(n\,;1,0) = \dim \widetilde{E}_k(-n\,;0,1) = 2(n+k)+1 \quad \text{for } k \geq 0. \tag{A.37}$$

Note that

$$\begin{aligned} \tilde{\lambda}_0(\pm n\,;0,0) &= n, \\ \tilde{\lambda}_0(n\,;0,1) = \tilde{\lambda}_0(-n\,;1,0) &= -n, \\ \tilde{\lambda}_0(n\,;1,0) - \tilde{\lambda}_0(-n\,;0,1) &= n. \end{aligned}$$

We now return to the second variation of the Yang-Mills functional YM. Similarly, we are also able to compute the spectrum of the elliptic operator

$$D_A^* D_A + D_A D_A^* + *[*F_A, \cdot] \quad \text{on } \Gamma(\operatorname{Ad}\eta(m,n) \otimes T^* S^2 \times S^2)$$

by separation of variables. But we are interested in finding isolated solutions to the Yang-Mills equations on $S^2 \times S^2$ which are nonminimal and reducible. By separation of variables we are able to prove that there exists a double indexing family of reducible nonminimal solutions to the Yang-Mills equations with group $SU(2)$ on $S^2 \times S^2$ which are isolated solutions.

Proposition A.4. *Let $L \to S^2$ be a tautological line bundle over S^2 and let A be the canonical connection on L. Suppose that S^2 admits the* *standard metric. Then we can construct a double indexing family of reducible nonminimal solutions to the Yang-Mills equations with group* SU(2) *on* $S^2 \times S^2$ *which are isolated:*

Choose a pair of integers (m, n) *such that*

(1) $|m| \neq |n|$.
(2) *If* $|m| > |n|$, *then* $|m| \neq |n|(2k+1) + k(k+1)$ *for* $k \geq 0$.
(3) *If* $|n| > |m|$, *then* $|n| \neq |m|(2k+1) + k(k+1)$ *for* $k \geq 0$.

Set $L(m, n) = \pi_1^* L^m \otimes \pi_2^* L^n$, *which is a line bundle over* $S^2 \times S^2$.

Put $A(m, n) = \pi_1^* \otimes^m A \otimes \pi_2^* \otimes^n A$, *which is a reducible connection on* $L(m, n) \oplus L(m, n)^{-1} \to S^2 \times S^2$ *with the second Chern number* $C_2 = -2mn$. *Then* $A(m, n)$ *is a reducible nonminimal solution to the Yang-Mills equations with group* SU(2) *on* $S^2 \times S^2$ *which is isolated.*

Proof. It is required to prove that $A(m, n)$ is an isolated solution. Using Proposition A.3, it is easy to check that the elliptic operator $D_A^* D_A + D_A D_A^* + *[f * F_A, \cdot]$ has no null eigenspace by separation of variables.

Remark A.3. As for a generic 4-manifold M, it is possible that there are no reducible self-dual or anti-self-dual connections over M. D. Freed and K. Uhlenbeck pointed out in [10] that if the intersection matrix of a 4-manifold is indefinite, then for an open dense metric set with which the 4-manifold is equipped, there are no line bundle solutions to the self-dual or anti-self-dual equations.

Now we consider the $S^1 \times S^3$. It is well known that S^3 is a homogeneous space whose Riemannian curvature is a constant. Hence the Levi-Civita connection A_0 on the tangent bundle TS^3 is a Yang-Mills connections over S^3 with structure group SO(3). In fact, the curvature F_{A_0} is parallel. Since SU(2) is the double covering of SO(3), it is easy to get an SU(2)-connection $\widetilde{A}_0$ over S^3 by lifting A_0, where $\widetilde{A}_0$ is also a parallel Yang-Mills connection. Let $A = \pi^* \widetilde{A}_0$. Here π is the projection $S^1 \times S^2 \to S^3$. Thus A is an irreducible nonminimal SU(2)-connection with degree $C_2(A) = 0$ over $S^1 \times S^3$. Using the analogous argument as in Proposition A.4, we are able to demonstrate that A is isolated. By separation of variables, it is required that A_0 be an isolated Yang-Mills connection over S^3. Bourguignon and Lawson in [6] have given a good description of this isolation phenomena. Their results are that the Levi-Civita connection on TS^3, or on $T(S^3/\Gamma)$, which are nontrivial quotients of S^3, is unstable as a Yang-Mills field. In fact its index is 1 and its nullity is 0 (cf. Theorem 9.2 in [6]). Hence, we have the following proposition.

Proposition A.5. *The Levi-Civita connection on* TS^3 *is an irreducible nonminimal Yang-Mills* SU(2)*-connection with degree zero over* $S^1 \times S^3$*, which is isolated.*

Appendix B. The power series expansion in the parameter λ

In §3 we defined a set of connections N_0 which are approximate solutions to the Yang-Mills equations. This appendix will be devoted to expanding the Yang-Mills functional YM on N_0 in the power series in the parameter λ.

Suppose $y \in N_0$. Then $A(y)$ is defined as

(B.1)

$$A(y) = \begin{cases} A(m, n) \quad \text{over } S^2 \times S^2, \\ \Gamma + \sum_{i=1}^{2k} \beta_r(x - q_i)\phi_i^* g_i W_i^2 g_i^{-1} + a \quad \text{over } V_0 \backslash \bigcup_{i=1}^{2k} V_i, \\ \Gamma + h_i[\phi_i^* g_i W_i^2 g_i^{-1} + (1 - \beta_{\lambda_i}(x - q_i)) \\ \qquad \cdot (\sum_{j \neq i} \beta_r(x - q_j)\phi_j^* g_j W_j^2 g_j^{-1} + a)]h_i^{-1} + h_i dh_i^{-1} \\ \qquad\qquad \text{over } V_i, \ 1 \le i \le 2k, \\ \Gamma + \phi_i^* g_i W_i^1 g_i^{-1} \quad \text{over } U_i, \ 1 \le i \le 2k. \end{cases}$$

Here $\lambda_i = s_i d^2$, $d = \pi/k$, $(W_i^1, W_i^2) = (W_{\lambda_i \pm}^1, W_{\lambda_i \pm}^2)$ as i is odd or even, and h_i is the gauge transformation

$$h_i\Big(\sum_{j \neq i} \beta_r(x - q_j)\phi_j^* g_j W_j^2 g_j^{-1} + a\Big)h_i^{-1} + h_i dh_i^{-1} \equiv \alpha_i, \tag{B.2}$$

which obeys

$$\alpha_j(q_i) = 0, \qquad \frac{\partial}{\partial |x - q_i|}\alpha_i = 0. \tag{B.3}$$

The above expressions were worked out in §3. It is not difficult to compute the corresponding curvature over the following domains

(1) over $S^2 \times S^2 \backslash V_0$:

$$F_{A(y)} = F(m, n);$$

(2) over $V_0 \setminus \bigcup_{i=1}^{2i} V_i$:

$$F_{A(y)} = F(m, n) + \sum_{i=1}^{2k} \beta_r(x - q_i)\phi_i^* g_i F_i^2 j g_i^{-1}$$
$$+ \sum_{i=1}^{2k} d\beta_r(x - q_i) \wedge \phi_i^* g_i W_i^2 g_i^{-1}$$
$$+ \sum_{i=1}^{2k} \beta_r(x - q_i)(1 - \beta_r(x - q_i))\phi_i^* g_i W_i^2 \wedge W_i^2 g_i^{-1}$$
$$+ \sum_{i \neq 1} \beta_r(x - q_i)\beta_j(x - q_j)\phi_i^* g_i W_i^2 g_i^{-1} \wedge \phi_j^* g_j W_j^2 g_j^{-1}$$
$$+ \sum_{i=1}^{2k} [a, \beta_r(x - q_i)\phi_i^* g_i W_i^2 g_i^{-1}];$$

(3) over V_i, $1 \le i \le 2k$:

$$F_{A(y)} = h_i \phi_i^* g_i F_i^2 g_i^{-1} h_i^{-1} + dh_i \wedge \phi_i^* g_i W_i^2 g_i^{-1} h_i^{-1}$$
$$+ h_i \phi_i^* g_i W_i^2 g_i^{-1} \wedge dh_i^{-1} + [h_i \phi_i^* g_i W_i^2 g_i^{-1} h_i^{-1}, \alpha_i] + \alpha_i \wedge \alpha_i;$$

(4) over U_i, $1 \le i \le 2k$:

$$F_{A(y)} = \phi_i^* g_i F_i^1 g_i^{-1}.$$

For simplicity, set $F(m, n) = F$, $\phi_i^* g_i W_i g_i^{-1} = W_i$, and $\phi_i^* g_i F_i g_i^{-1} = F_i$. Thus, we have

$$\text{(B.4)} \qquad \mathrm{YM}(A(y)) = \tfrac{1}{2} \int |F_{A(y)}|^2 = (I_1) + (I_2) + (I_3) + (I_4) + (I_5),$$

where

$$(I_1) = \frac{1}{2} \int \left| \prod_{i=1}^{2k} (1 - \beta_{\lambda_i}(x - q_i))F + d\left\{ \prod_{i=1}^{2k} (1 - \beta_{\lambda_i}(x - q_i)) \right\} \wedge a \right.$$

$$\text{(B.5)} \qquad + \sum_{\text{odd } i} \left\{ \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i)F_k^2 \right.$$
$$+ \prod_{l=1}^{2k} (1 - \beta_{\lambda_l}(x - q_l))a \wedge \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i)W_i^2$$
$$+ \left(1 - \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i)\right)$$

$$\times \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2\wedge W_i^2$$

$$+\sum_{\substack{i\neq l\\ \text{odd } i,l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))$$

$$\cdot\beta_r(x-q_l)W_i^2\wedge W_l^2$$

$$+d\left\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right\}\wedge W_i^2\Big\}\Big|^2,$$

(B.6)

$$(I_2)=\frac{1}{2}\int\Big|\prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))F+d\left\{\prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))\right\}\wedge a$$

$$+\sum_{\text{even } i}\Big\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i^2$$

$$+\prod_{l=1}^{2k}(1-\beta_{\lambda_l}(x-q_l))a\wedge\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2$$

$$+\left(1-\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))$$

$$\cdot\beta_r(x-q_i)W_i^2\wedge W_i^2$$

$$+\sum_{\substack{i\neq l\\ \text{even } i,l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))$$

$$\cdot\beta_r(x-q_l)W_i^2\wedge W_l^2$$

$$+d\left\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right\}\wedge W_i^2\Big\}\Big|^2,$$

(B.7)

$$(I_3)=\int\Big\langle\sum_{\text{odd } i}\Big\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i^2$$

$$+\prod_{l=1}^{2k}(1-\beta_{\lambda_l}(x-q_l))a\wedge\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2$$

$$+\left(1-\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))$$
$$\cdot\beta_r(x-q_i)W_i^2\wedge W_i^2$$
$$+\prod_{\substack{i\neq l\\ \text{odd } i,l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))$$
$$\cdot\beta_r(x-q_l)W_i^2\wedge W_l^2$$
$$+d\left\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right\}\wedge W_i^2\Bigg\},$$

$$\sum_{\text{even } i}\Bigg\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i^2$$
$$+\prod_{i=1}^{2k}(1-\beta_{\lambda_l}(x-q_l))a\wedge\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2$$
$$+\left(1-\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))$$
$$\cdot\beta_r(x-q_i)W_i^2\wedge W_i^2$$
$$+\prod_{\substack{i\neq l\\ \text{even } i,l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))$$
$$\cdot\beta_r(x-q_l)W_i^2\wedge W_l^2$$
$$+d\left\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right\}\wedge W_i^2\Bigg\}\Bigg\rangle,$$

(B.8)

$$(I_4)=\int\Bigg\langle F,\sum_{\substack{i\neq l\\ \text{odd } i\\ \text{even } l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)$$
$$\cdot\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l)W_i^2\wedge W_l^2\Bigg\rangle,$$

(B.9)

$$(I_5) = -\frac{1}{2}\int \left| \prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))F + d\left\{ \prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i)) \right\} \wedge a \right|^2 .$$

In order to expand the Yang-Mills functional $\mathrm{YM}(A(y))$ in the power series in the parameter λ, one requires the following estimates. Identify $\mathscr{R}^4 \cong \mathscr{H}$ = quaternions, $\mathrm{SU}(2) \cong$ unit quaternions, and $L(\mathrm{SU}(2)) = \mathrm{Im}\,\mathscr{H}$. On $\mathscr{R}^4$, define

$$x = x^0 + x^1T^1 + x^2T^2 + x^3T^3,$$

where $\{T^\alpha\}_{1\le\alpha\le3}$ is the orthonormal basis of $L(\mathrm{SU}(2))$. Thus

$$\begin{aligned} d\overline{x}\wedge dx = 2\{&(dx^0\wedge dx^1 - dx^2\wedge dx^3)T^1 \\ &+ (dx^0\wedge dx^2 - dx^3\wedge dx^1)T^2 \\ &+ (dx^0\wedge dx^3 - dx^1\wedge dx^3)T^3\}, \end{aligned} \tag{B.10}$$

$$\begin{aligned} dx\wedge d\overline{x} = -2\{&(dx^0\wedge dx^1 + dx^2\wedge dx^3)T^1 \\ &+ (dx^0\wedge dx^2 + dx^3\wedge dx^1)T^2 \\ &+ (dx^0\wedge dx^3 + dx^1\wedge dx^2)T^3 . \end{aligned} \tag{B.11}$$

Set

$$\begin{aligned} \omega^\alpha &= (2\sqrt{2})^{-1}(T^\alpha, d\overline{x}\wedge dx) \\ &= (2\sqrt{2})^{-1}R_e(T^\alpha, \overline{d\overline{x}\wedge dx}) \quad \text{for } \alpha = 1, 2, 3, \end{aligned} \tag{B.12}$$

$$\begin{aligned} \overline{\omega}^\alpha &= (-2\sqrt{2})^{-1}(T^\alpha, dx\wedge d\overline{x}) \\ &= (-2\sqrt{2})^{-1}R_e(T^\alpha, \overline{dx\wedge d\overline{x}}) \quad \text{for } \alpha = 1, 2, 3. \end{aligned} \tag{B.13}$$

Since $A(m, n)$ is a reducible SU(2)-connection over $S^2\times S^2$, by the construction of $A(m, n)$ we obtain

$$F(m, n) = \frac{1}{2}(m\omega_1 + n\omega_2)\begin{pmatrix} i & o \\ o & -i \end{pmatrix}, \tag{B.14}$$

where ω_1 and ω_2 are volume forms. It is easy to compute that

$$
\begin{aligned}
|x|\frac{\partial}{\partial|x|}\lrcorner\,\omega^\alpha &= (2\sqrt{2})^{-1}|x|\frac{\partial}{\partial|x|}\lrcorner\,(T^\alpha, d\bar{x}\wedge dx) \\
&= \frac{1}{\sqrt{2}}(T^\alpha, \bar{x}dx), \\
|x|\frac{\partial}{\partial|x|}\lrcorner\,\overline{\omega}^\alpha &= (-2\sqrt{2})^{-1}|x|\frac{\partial}{\partial|x|}\lrcorner\,(T^\alpha, dx\wedge d\bar{x}) \\
&= -\frac{1}{\sqrt{2}}(T^\alpha, xd\bar{x}).
\end{aligned}
\tag{B.15}
$$

Recalling the expressions of $W^2_{\lambda+}$, we have

$$
W^2_{\lambda+} = \lambda^*\vartheta^*(W^1_-), \tag{B.16}
$$

where $W^1_- = \bar{x}dx/(1+|x|^2)$ is the standard anti-self-dual SU(2)-connection over $\mathscr{R}^4$, λ^* denotes scaling, i.e., $\lambda^*(x) = x/\lambda$, and ϑ is the inversion for $\mathscr{R}^4$. Since

$$
\begin{gathered}
F_- = dW^1_- + W^1_-\wedge W^1_- = \frac{d\bar{x}\wedge dx}{(1+|x|^2)^2} = d\bar{x}\wedge dx + R(x), \\
|R(x)| \le \frac{2|x|^2+|x|^4}{(1+|x|^2)^4}|d\bar{x}\wedge dx|,
\end{gathered}
$$

and

$$
\begin{aligned}
\vartheta^*(d\bar{x}\wedge dx) &= d\left(\frac{\bar{x}}{|x|^2}\right)\wedge d\left(\frac{x}{|x|^2}\right) \\
&= \frac{-\bar{x}dx\bar{x}}{|x|^4}\wedge\frac{-xd\bar{x}x}{|x|^4} = |x|^{-4}(d\bar{x}\wedge dx),
\end{aligned}
$$

we have

$$
\vartheta^*\omega^\alpha = |x|^{-4}\omega^\alpha. \tag{B.17}
$$

Using (B.10)–(B.17) yields

$$
\begin{aligned}
W^2_{\lambda+} = \lambda^*\vartheta^*W^1_- &= \lambda^*\vartheta^*\int_0^1 d\tau\tau|z|\frac{\partial}{\partial|z|}\lrcorner\,F^1_-(\tau z) \\
&= \lambda^*\vartheta^*\int_0^1 d\tau\left\{\tau|z|\frac{\partial}{\partial|z|}\lrcorner\,(F^1_-(N)+R)\right\},
\end{aligned}
$$

where

$$\begin{aligned}
\lambda^*\vartheta^* &\int_0^1 d\tau\tau|z|\frac{\partial}{\partial|z|}F_-(N) \\
&= \frac{1}{2\sqrt{2}}\lambda^*\vartheta^*\{(F_-(N), \omega^\alpha)(T^\alpha, \overline{z}dz)\} \\
&= \frac{1}{2\sqrt{2}}(F_-(N), \omega^\alpha)\lambda^*\left\{\left(T^\alpha, \frac{\overline{z}}{|z|^2}d\left(\frac{z}{|z|^2}\right)\right)\right\} \\
&= \frac{-1}{2\sqrt{2}}(F_-(N), \omega^\alpha)\left(T^\alpha, \lambda^*\frac{d\overline{z}z}{|z|^4}\right) \\
&= \frac{\lambda^2}{2\sqrt{2}}|x|^{-4}(F_-(N), \omega^\alpha)(T^\alpha, \overline{x}dx),
\end{aligned}$$

$$\begin{aligned}
|\lambda^*\vartheta^* &\int_0^1 d\tau\tau|z|\frac{\partial}{\partial|z|}\rfloor R(z)| \\
&\leq \lambda^*\vartheta^* C|z|^3|dz| = \lambda^* C\frac{|dz|}{|z|^5} = C\lambda^4|dx|\,|x|^{-5}.
\end{aligned}$$

Therefore

$$\begin{aligned}
W_{\lambda+}^2 &= \frac{\lambda^2}{2\sqrt{2}}|x|^{-4}(F_-(N), \omega^\alpha)(T^\alpha, \overline{x}dx) + R, \\
|R| &\leq C\lambda^4|dx|\,|x|^{-5}.
\end{aligned} \tag{B.18}$$

Suppose that $\eta \in C_0^\infty(\mathscr{R})$ is a radial function; then

$$\begin{aligned}
d\eta \wedge &(T^\alpha, \overline{x}dx) \\
&= \frac{1}{2}\frac{\partial\eta}{\partial|x|}\frac{d|x|^2}{|x|} \wedge (T^\alpha, \overline{x}dx) \\
&= \frac{1}{4}|x|^{-1}\frac{\partial\eta}{\partial|x|}(T^\alpha, d|x|^2 \wedge \overline{x}dx - \overline{x}dx \wedge d|x|^2) \\
&= \frac{1}{4}|x|^{-1}\frac{\partial\eta}{\partial|x|}(T^\alpha, (d\overline{x}x + \overline{x}dx) \wedge \overline{x}dx - \overline{x}dx \wedge (\overline{x}dx + d\overline{x}x)) \\
&= \frac{1}{4}|x|\frac{\partial\eta}{\partial|x|}(T^\alpha, d\overline{x} \wedge dx - |x|^{-2}\overline{x}dx \wedge d\overline{x}x).
\end{aligned}$$

Hence, we have

$$P_- d\eta \wedge (T^\alpha, \overline{x}dx) = \frac{1}{2}|x|\frac{\partial\eta}{\partial|x|}\omega^\alpha, \tag{B.19}$$

$$
\begin{aligned}
&P_+\, d\eta \wedge (T^\alpha , \overline{x}dx) \\
&\quad = \frac{1}{4}|x|\frac{\partial \eta}{\partial |x|}(T^\alpha , |x|^{-2}\overline{x}dx \wedge d\overline{x}x) \\
&\quad = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}(T^\alpha , |x|^{-2}\overline{x}T^\beta x)\overline{\omega}^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|^{-1}\frac{\partial \eta}{\partial |x|}(T^\alpha , T^\beta (x - 2x^\beta T^\beta)x)\overline{\omega}^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|^{-1}\frac{\partial \eta}{\partial |x|}(T^\alpha , 2x^\beta x + T^\beta x^2)\overline{\omega}^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|^{-1}\frac{\partial \eta}{\partial |x|}(T^\alpha , 2x^\beta x + T^\beta (2x^0 x - |x|^2))\overline{\omega}^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}|x|^{-2}\Bigg\{\Bigg(|x^0|^2 + |x^\alpha|^2 - \sum_{\substack{1\le\gamma\le 3\\ \gamma\ne\alpha}}|x^\gamma|^2\Bigg)\overline{\omega}^\alpha \\
&\qquad\qquad + \sum_{\substack{\beta\ne\alpha\\ 1\le\beta\le 3}}(2x^\beta x^\alpha + 2\delta^{\alpha\beta\gamma}x^0 x^\gamma)\overline{\omega}^\beta\Bigg\}.
\end{aligned} \tag{B.20}
$$

Similarly,

$$
\begin{aligned}
W_{\lambda_-}^2 &= \frac{\lambda^2}{2\sqrt{2}}|x|^{-4}(F_+(N), \overline{\omega}^\alpha)(T^\alpha , xd\overline{x}) + R, \\
|R| &\le C\lambda^4 |dx|\,|x|^{-5},
\end{aligned} \tag{B.21}
$$

$$
-d\eta\wedge(T^\alpha , xd\overline{x}) = -\frac{1}{4}|x|\frac{\partial \eta}{\partial |x|}(T^\alpha , dx\wedge d\overline{x} - \frac{x}{|x|^{-2}}\, d\overline{x}\wedge dx\overline{x}), \tag{B.22}
$$

$$
-P_+\, d\eta \wedge (T^\alpha , xd\overline{x}) = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}\overline{\omega}^\alpha , \tag{B.23}
$$

$$
\begin{aligned}
&- P_-\, d\eta \wedge (T^\alpha , xd\overline{x}) \\
&\quad = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}(T^\alpha , xT^\beta \overline{x})|x|^{-2}\omega^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}|x|^{-2}(T^\alpha , -2x^\beta \overline{x} + T^\beta (x^0\overline{x} - |\overline{x}|^2))\omega^\beta \\
&\quad = \frac{1}{\sqrt{2}}|x|\frac{\partial \eta}{\partial |x|}|x|^{-2}\Bigg\{(|x^0|^2 + |x^\alpha|^2 - \sum_{\substack{1\le\gamma\le 3\\ \gamma\ne\alpha}}|x^\gamma|^2)\omega^\alpha \\
&\qquad\qquad + \sum_{\substack{1\le\beta\le 3\\ \beta\ne\gamma}}(2x^\beta x^\alpha - 2\delta^{\alpha\beta\gamma}x^0 x^\gamma)\omega^\beta\Bigg\}.
\end{aligned} \tag{B.24}
$$

Let g be the product metric on $S^2 \times S^2$ with S^2 of the radius 1. For simplicity we denote by $*_g$ and $*_e$ the Hodge operators on V_i, $1 \leq i \leq 2k$, with respect to the metric g and the Euclidean flat metric e respectively. We now use inequality (3.11) of §3 to obtain the estimate

(B.25)
$$|*_g - *_e| \leq O(K|x|^2),$$

where K is the scalar curvature on $S^2 \times S^2$, which is a positive constant.

By direct calculation, we obtain

(B.26)
$$\begin{aligned}(I_5) &= -\frac{1}{2}\int \left| \prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))F + d\left\{\prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))\right\}\Lambda a\right|^2 \\ &= -8\pi^2(m^2+n^2) + C\sum_{i=1}^{2k}\lambda_i^4|F(m,n)|^2,\end{aligned}$$

(B.27)
$$\begin{aligned}(I_1) &= 8\pi^2(m^2+n^2+k) \\ &\quad + \int \Bigg\langle P_-\left\{\prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))F + d\left(\prod_{i=1}^{2k}(1-\beta_{\lambda_i}(x-q_i))\right)\Lambda a\right\}, \\ &\qquad P_-\sum_{\text{odd } i}\Bigg\{\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i \\ &\qquad\qquad + \prod_{l=1}^{2k}(1-\beta_{\lambda_l}(x-q_l))a \\ &\qquad\qquad \wedge \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2 \\ &\qquad\qquad + \left(1-\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right) \\ &\qquad\qquad \cdot\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2\wedge W_i^2 \\ &\qquad\qquad + \sum_{\substack{l\neq i\\ \text{odd } l}}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i) \\ &\qquad\qquad \cdot\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l)W_i^2\wedge W_l^2 \\ &\qquad\qquad + d\left(\prod_{j\neq i}(-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)\wedge W_i^2\Bigg\}\Bigg\rangle\end{aligned}$$

$$+\int \Bigg| P_- \sum_{\text{odd } i} \Bigg\{ \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i$$

$$+\prod_{l=1}^{2k}(1-\beta_{\lambda_l}(x-q_l))a\wedge\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))$$

$$\cdot\,\beta_r(x-q_i)W_i^2$$

$$+\left(1-\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)$$

$$\cdot\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\,\beta_r(x-q_i)W_i^2\wedge W_i^w$$

$$+\sum_{\substack{l\neq i\\ \text{odd } l}}\prod_{j\neq i}(1-\beta_{\lambda_j}))\beta_r(x-q_i)\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))$$

$$\cdot\,\beta_r(x-q_l)W_i^2\wedge W_l^2$$

$$+d\left(\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right)\wedge W_i^2\Bigg\}\Bigg|^2.$$

Since

(B.28)

$$2\int\Bigg\langle P_-\Bigg\{\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))F+d\Bigg[\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))\Bigg]\wedge a\Bigg\},$$

$$P_-\sum_{\text{odd } i}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i\Bigg\rangle$$

$$=C\sum_{\text{odd } i}\Bigg\{|P\_F(m,n)|\lambda_i^2r^2+|P\_F(m,n)|\lambda_i^4$$

$$+|P\_F(m,n)|\lambda_i^2\sum_{j=1}^{[2r/d]}\frac{\lambda^4_{(i\pm j)\,\mathrm{mod}\;2k}}{j^2d^2}\Bigg\},$$

(B.29)

$$2\int\Bigg\langle P_-\Bigg\{\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))F+d\Bigg[\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))\Bigg]\wedge a\Bigg\},$$

$$P_- \sum_{\text{odd } i} \left\{ \prod_{l=1}^{2k} (1 - \beta_{\lambda_l}(x - q_l)) a_0 \right.$$

$$\left. \left. \wedge \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j)) + \beta_r(x - q_i) W_i^2 \right\} \right\rangle$$

$$= C \sum_{\text{odd } i} \left\{ |P\_F(m, n)|^2 \left(a_i^2 r^2 + \lambda_i^3 + \sum_{j=1}^{[2r/d]} \frac{\lambda^4_{(i \pm j) \bmod 2k}}{j^2 d^2} \right) \right\},$$

(B.30)

$$2 \int \left\langle P_- \left\{ \prod_{j=1}^{2k} (1 - \beta_{\lambda_j}(x - q_j)) F + d \left[\prod_{j=1}^{2k} (1 - \beta_{\lambda_j}(x - q_j)) \right] \wedge a \right\}, \right.$$

$$P_- \sum_{\text{odd } i} \left(1 - \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j)) \beta_r(x - q_i) \right)$$

$$\left. \cdot \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j)) \beta_r(x - q_i) W_i^2 \wedge W_i^w \right\rangle$$

$$= C \sum_{\text{odd } i} \left\{ |P\_F(m, n)| \left(\lambda_i^4 r^{-2} + \lambda_i^4 \sum_{j=[r/d]}^{[2r/d]} \frac{\lambda^4_{(i \pm j) \bmod 2k}}{r^6} \right) \right\},$$

and

(B.31)

$$2 \int \left\langle P_- \left\{ (1 - \beta_{\lambda_j}(x - q_j)) F + d \left[\prod_{j=1}^{2k} (1 - \beta_{\lambda_j}(x - q_j)) \right] \wedge a \right\}, \right.$$

$$P_- \sum_{\substack{\text{odd } l, i \\ l \neq i}} \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j)) \beta_r(x - q_i)$$

$$\left. \cdot \prod_{h \neq l} (1 - \beta_{\lambda_h}(x - q_h)) \beta_r(x - q_l) W_i^2 \wedge W_l^2 \right\rangle$$

$$= C \sum_{\text{odd } i} |P\_F(m, n)| \lambda_i^2 \sum_{l=1}^{[r/d]} \frac{r \lambda^2_{(i \pm j) \bmod 2k}}{(2ld)^3},$$

we have
(B.32)

$$2\int \left\langle P_-\left\{\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))F + d\left[\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))\right]\wedge a\right\},\right.$$
$$\left. P_-\sum_{\text{odd } i} d\left[\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right]\wedge W_i^2\right\rangle$$
$$= 2\int \left\langle P_-\left(\prod_{j=1}^{2k}(1-\beta_{\lambda_j}(x-q_j))F\right),\right.$$
$$\left. P_-\sum_{\text{odd } i}^{2k} d\left[\prod_{i=1}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right]\wedge W_i^2\right\rangle$$
$$+ C\sum_{\text{odd } i}|P\_F(m,n)|\left(\lambda_i^2 r^{-3}\sum_{j=[r/d]}^{[2r/d]}\lambda_{i\pm j}^4 + \lambda_i^2\sum_{j\neq i}\frac{\lambda_j^3}{|(j-i)d|^3}\right)$$
$$= 2\int \left\langle P_-\left(\prod_{j=1}(1-\beta_{\lambda_j}(x-q_j))F\right), P_-\sum_{\text{odd } i} d\beta_r(x-q_i)\wedge W_i^2\right\rangle$$
$$+ C\sum_{i \text{ odd}}|P\_F(m,n)|\left(\frac{\lambda_i^2}{r^3}\sum_{j=[r/d]}^{[2r/d]}\lambda_{(i\pm j)\bmod 2k}^4 + \lambda_i^2\sum_{j\neq i}\frac{2\lambda_j^2}{|(j-i)d|^3}\right)$$
$$= \sum_{\text{odd } i} -\frac{\omega(S^3)}{2}\langle P\_F(m,n)(q_i), \phi_i^* g_i F_-(N)g_i^{-1}\rangle\lambda_i^2$$
$$+ C\sum_{\text{odd } i}|P\_F(m,n)|\left(\lambda_i^2 r^2 + \lambda_i^2 r^{-3}\sum_{j=[r/d]}^{[2r/d]}\lambda_{(i\pm j)\bmod 2k}^4\right.$$
$$\left. + \lambda_i^2\sum_{j\neq i}\frac{2\lambda_j^2}{|(j-i)d|^3}\right),$$

where $\omega(S^3)$ denotes the volume of unit 3-sphere S^3. In the above estimates we have used (B.18) and (B.19). We now give the estimates of other terms:

(B.33)
$$\int\left|P_-\sum_{\text{odd } i}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i\right|^2$$
$$= \sum_{\text{odd } i} C_1\lambda_k^4|\ln\lambda_i| + \sum_{\substack{i\neq l\\ \text{odd } i,l}} C_2\lambda_i^2\lambda_k^2|(l-i)d|^{-2},$$

(B.34)
$$\begin{aligned}\int \Bigg| P_- \sum_{\text{odd } i} \prod_{l=1}^{2k} (1-\beta_{\lambda_l}(x-q_l))a \wedge \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j)) \\ \cdot \beta_r(x-q_i)W_i^2\Bigg|^2 \\ = \sum_{i \text{ odd}} C_1 |P\_F(m, n)| \lambda_i^4 | \ln \lambda_i | \\ + \sum_{\substack{i\neq l \\ \text{odd } i, l}} C_2 |P_-(m, n)| \lambda_i^2 \lambda_l^2 r^2 |(l-i)\, d|^{-2},\end{aligned}$$

(B.35)
$$\begin{aligned}\int \Bigg| P_- \Bigg\{ \Bigg(1 - \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\Bigg) \\ \cdot \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)W_i^2 \wedge W_i^2 \Bigg\}\Bigg|^2 \\ = k \sum_{\text{odd } i} C \int |\beta_r(1-\beta_r) W_i^2 W_i^2|^2 \\ = k \sum_{\text{odd } i} C\lambda_i^8 r^{-8} = \sum_{\text{odd } i} C\lambda_i^8 r^{-7} d^{-1},\end{aligned}$$

(B.36)
$$\begin{aligned}\int \Bigg| P_- \sum_{\substack{\text{odd } l, i \\ l\neq i}} \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i) \\ \cdot \prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l) W_i^2 \wedge W_l^2 \Bigg|^2 \\ = k(k-1) \sum_{\substack{\text{odd } l, i \\ j\neq i}} C \int |\beta_r(x-q_i)\beta_r(x-q_l) W_i^2 \wedge W_l^2|^2 \\ = k(k-1) \sum_{\substack{\text{odd } l, i \\ l\neq i}} C\lambda_i^4 \lambda_k^4 r^{-2} \frac{1}{|(l-i)\, d|^6} \\ = \sum_{\substack{\text{odd } l, i \\ l\neq i}} C\lambda_i^4 \lambda_l^4 |l-i|^{-6} d^{-8} r^{-2},\end{aligned}$$

(B.37)

$$\int \left| P_- \sum_{\text{odd } i} d\left[\prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i) \right] \wedge W_i^2 \right|^2$$

$$= \sum_{\text{odd } i} \left(C_1 \lambda_i^4 r^2 + C_2 \lambda_i^2 r \sum_{l=1}^{2k} \lambda_l^2 \sum_{l=1}^{2k} \frac{\lambda_j^3}{\lambda^3 d^3} + C_3 2k \lambda_i^2 \sum_{l=1}^{2k} \frac{\lambda_l^3}{j^3 d^3} \right).$$

Hence, we have the following estimates of (I_1):

$$(I_1) = 8\pi^2(m^2 + n^2 + k)$$

$$+ \sum_{\text{odd } i} -\frac{\omega(S^3)}{2} \lambda_i^2 \langle P\_F(m, n)(q_i), \phi_i^* g_i F_-(N) g_i^{-1} \rangle \tag{B.38}$$

$$+ \sum_{\text{odd } i} (C|P\_F(m, n)| \lambda_i^2 r^2 + \text{higher order terms}).$$

Similarly,

$$(I_2) = 8\pi^2(m^2 + n^2 + k)$$

$$+ \sum_{\text{even } i} -\frac{\omega(S^3)}{2} \lambda_i^2 \langle P_+ F(m, n)(q_i), \phi_i^* g_i F_+(N) g_i^{-1} \rangle \tag{B.39}$$

$$+ \sum_{\text{even } i} (C|P_+ F(m, n)| \lambda_k^2 r^2 + \text{higher order terms}).$$

We now estimate every term in (I_3):

$$\int \left\langle \sum_{\text{odd } i} P_+ \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i) F_i, \right.$$

$$\left. \sum_{\text{even } l} P_- \prod_{h \neq l} (1 - \beta_{\lambda_h}(x - q_h))\beta_r(x - q_l) F_l \right\rangle \tag{B.40}$$

$$= C \sum_{\text{odd } i} \lambda_i^2 r^2 + \text{higher order terms},$$

$$\int \left\langle \sum_{\text{odd } i} P_- \prod_{j \neq i} (1 - \beta_{\lambda_j}(x - q_j))\beta_r(x - q_i) F_i, \right.$$

$$\sum_{\text{even } l} P_- \prod_{l=1}^{2k} (1 - \beta_{\lambda_l}(x - q_l)) a \tag{B.41}$$

$$\left. \wedge \prod_{h \neq l} (1 - \beta_{\lambda_h}(x - q_h))\beta_r(x - q_l) W_l^2 \right\rangle$$

$$= C \sum_{\text{odd } i} |F(m, n)| \lambda_i^2 d^2 r^2 + \text{higher order terms},$$

(B.42)
$$\begin{aligned}\int \Big\langle \sum_{\text{odd } i} P_- \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q)F_i\,,\\ \sum_{\text{even } l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_i)\\ \cdot\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l)W_l^2\wedge W_h^2\Big\rangle\\ =\sum_{\text{odd } i}\sum_{\text{even } l} C\lambda_i^2\lambda_l^4 r^{-4} + \text{higher order terms},\end{aligned}$$

(B.43)
$$\begin{aligned}\int \Big\langle \sum_{\text{odd } i} P_- \prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i\,,\\ \sum_{\substack{h\neq l\\ \text{even } h,\,l}}\prod_{j\neq h}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_h)\\ \cdot\prod_{j\neq l}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-ql)W_h^2\wedge W_l^2\Big\rangle\\ =C\sum_{\text{odd } i}\sum_{\text{even } h,\,l}\lambda_i^2\lambda_h^2\lambda_l^2 r^2 d^{-6} + \text{higher order terms},\end{aligned}$$

(B.44)
$$\begin{aligned}\int \Big\langle P_- \sum_{\text{odd } i}\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)F_i\,,\\ \sum_{\text{even } l} d\left[\prod_{j\neq l}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_l)\right]\wedge W_l^2\Big\rangle\\ =\sum_{\text{odd } i}\lambda_i^2\{C_1 r^{-1}d^3 + C_2\lambda_i d + C_3 d^5\},\end{aligned}$$

(B.45)
$$\begin{aligned}\int \Big\langle P_- \sum_{\text{odd } i} d\left[\prod_{j\neq i}(1-\beta_{\lambda_j}(x-q_j))\beta_r(x-q_i)\right]\wedge W_i^2\,,\\ P_-\sum_{\text{even } l}\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l)F_l\Big\rangle\\ =C_1\sum_{\text{odd } i}\sum_{h=0}^{[r/d]}\lambda_i^2\lambda^2_{(i\pm 2h)\ \mathrm{mod}\ 2k}|\ln\lambda_{(i\pm 2h)\ \mathrm{mod}\ 2k}|r^{-4}\\ +\int\Big\langle\sum_{\text{odd } i}\sum_{h\neq l}P_-\,d(1-\beta_{\lambda_h}(x-q_h))\wedge W_i^2\beta_r(x-q_i)\,,\\ \sum_{\text{even } l}\prod_{h\neq l}(1-\beta_{\lambda_h}(x-q_h))\beta_r(x-q_l)F_l\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\text{odd } i} C_1 \lambda_i^2 d^3 |\ln d| r^{-3} \\
&\quad + \sum_{\text{odd } i} \sum_{\text{even } l} C_2 \lambda_i^2 \lambda_h^2 \sum_{h \neq l, i} \lambda_h^3 \operatorname{dist}(q_i, q_h)^{-3} \operatorname{dist}(q_l, q_h)^{-4} \\
&\quad + \sum_{\text{odd } i} \sum_{\text{even } l} \lambda_i^2 \lambda_l^2 \int \left\langle P_- \frac{\beta_r(x-q_i)\, d\beta_r(x-q_l) \wedge \overline{(x-q_i)}\, d(x-q_i)}{\operatorname{dist}(q_i, q_l)^4}, \right. \\
&\qquad\qquad\qquad \left. \frac{d\overline{(x-q_l)} \wedge d(x-q_l)}{\lambda_l^2 + \operatorname{dist.}(x, q_l)^2)^2} \right\rangle \\
&\quad + \sum_{\text{odd } i} \sum_{\text{even } l} \lambda_i^2 \lambda_l^2 \int_{\beta_r(x-q_i) \cap \beta_r(x-q_l)} \frac{\operatorname{dist}(x-q_l)^2}{\operatorname{dist}(q_i, q_l)^3 (\lambda_l^2 + \operatorname{dist}(x, q_l)^2)^2 \lambda_l} \\
&= \sum_{\text{odd } i} (C_1 \lambda_i^2 d^3 |\ln d| r^{-3} + C_2 \lambda_2 \lambda_i^2 d^2) + \sum_{\text{odd } i} \sum_{\text{even } l} \lambda_i^2 \lambda_l^2 \frac{1}{\operatorname{dist}(q_i, q_l)^4} \\
&\qquad \cdot \int \langle P_- \beta_r(x-q_i)\, d\beta_{\lambda_l}(x-q_l) \wedge \overline{(x-q_i)} \wedge d(x-q_l),\, P_- F_l \rangle \\
&= \sum_{\text{odd } i} (C_1 \lambda_i^2 |\ln d| r^{-3} + C_2 \lambda_i^2 d^2) + \sum_{\text{odd } i} \sum_{\text{even } l} \lambda_i^2 \lambda_l^2 \frac{1}{\operatorname{dist}(q_i, q_l)^4} \\
&\qquad \cdot \int \left\langle P_- \beta_r(x-q_i)\, d\beta_{\lambda_l}(x-q_l) \wedge \overline{(x-q_l)}\, d(x-q_l), \right. \\
&\qquad\qquad\qquad \left. P - \frac{d\overline{(x-q_l)} \wedge d(x-q_l)}{(\lambda_l^2 + \operatorname{dist.}(x-q_l)^2)^2} \right\rangle \\
&= \sum_{\text{odd } i} (C_1 \lambda_i^2 d^3 |\ln d| r^{-3} + C_2 \lambda_i^2 d^2 + C_3 \lambda_i^2 d^3 r^{-3}) \\
&\quad + \sum_{\text{odd } i} \sum_{\substack{\text{even } l \\ l = [i \pm (2h+1)] \bmod 2k \\ 0 \le h \le [r/2d]-1}} -Q \frac{\omega(S^3)}{2} \frac{\lambda_i^2 \lambda_l^2}{\operatorname{dist}(q_i, q_l)^4} \\
&\qquad \cdot \langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_l^* g_l F_-(N) g_l^{-1} \rangle,
\end{aligned}
$$

where $Q = -\int_1^2 \partial \beta r^4\, dr/(1+r^2)^2 > 0$ is a constant.

Recall the construction of $A(y)$. Over V_i, $1 \le i \le 2k$, we define

$$A(y) = \Gamma + h_i\Big[\phi_i^* g_i W_i^2 g_i^{-1} + (1-\beta_{\lambda_i}(x-q_i)) \cdot \Big(\sum_{j\neq i} \beta_r(x-q_j)\phi_j^* g_j W_j^2 g_j^{-1} + a\Big] h_i^{-1} + h_i\, dh_i^{-1}\Big),$$

where $\lambda_i = s_i d^2$, $d = \pi/d$, $W_i^2 = W_{\lambda_i \pm}^2$ according as i is odd or even, and h_i is the gauge transformation

$$h_i\Big(\sum_{j\neq i} \beta_r(x-q_j)\phi_j^* g_j W_j^2 g_j^{-1} + a\Big) h_i^{-1} + h_i\, dh_i^{-1} \equiv \alpha_i,$$

which obeys

$$\alpha_i(q_i) = 0, \qquad \frac{\partial}{\partial|x-q_i|} \rfloor \alpha_i = 0.$$

Hence α_i is the polar gauge potential. By direct calculation, we find that the other terms of (I_3) are higher order terms. From this, we have the following estimates for (I_3):

$$\begin{aligned}(I_3) = &\sum_{\text{odd } i} \sum_{\substack{\text{even } l\\ l=[i\pm(2h+1)] \bmod 2k \\ 0\leq h\leq [r/2d]-1}} -Q\frac{\omega(S^3)}{2} \frac{\lambda_i^2\lambda_l^2}{\operatorname{dist}(q_i, q_l)^4} \\ &\cdot\langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_l^* g_l F_-(N) g_l^{-1}\rangle \\ &+ \sum_{\text{odd } i} \sum_{\substack{\text{odd } l\\ l=[i\pm(2h+1)] \bmod 2k \\ 0\leq h\leq [r/2d]-1}} -Q\frac{\omega(S^3)}{2} \frac{\lambda_k^2\lambda_l^2}{\operatorname{dist}(q_i, q_l)^4} \\ &\cdot\langle \phi_i^* g_i F_+(N) g_i^{-1}, \phi_l^* g_l F_+(N) g_l^{-1}\rangle \\ &+ \sum_{i=1}^{2k} (C_1 \lambda_i^2 d^3 |\ln d| r^{-3} + C_2 \lambda_i^2 r^2 + C_3 \lambda_i d^3 r^{-3} \\ &\qquad + \text{higher order terms}).\end{aligned} \tag{B.46}$$

In the same way we may estimate (I_4), which is higher order terms. In terms of the above computations, we obtain the following proposition immediately.

Proposition B.1. *If the Yang-Mills functional is restricted on the space* N_1, *we have a power series expansion of the Yang-Mills functional* *in parameter* λ:

(B.47)

$$\begin{aligned}\mathrm{YM}(A(y)) &= \frac{1}{2}\int |F_{A(y)}|^2 \\ &= 8\pi^2(m^2+n^2+2k) \\ &\quad + \sum_{\text{odd } i} -\frac{\omega(S^3)}{2}\lambda_i^2\langle P_-F(m,n)(q_i), \phi_i^* F_-(N) g_i^{-1}\rangle \\ &\quad + \sum_{\text{even } j} -\frac{\omega(S^3)}{2}\lambda_j^2\langle P_+(m,n)(q_i), \phi_j^* g_j F_+(N) g_j^{-1}\rangle \\ &\quad + \sum_{\text{odd } i} \sum_{\substack{\text{even } j \\ j=[i\pm(1l+1)] \bmod 2k \\ 0\le l\le [r/2d]-1}} -Q\frac{\omega(S^3)}{2}\frac{\lambda_i^2\lambda_j^2}{\mathrm{dist}(q_i,q_j)^4} \\ &\quad \cdot\langle \phi_i^* g_i F_-(N) g_i^{-1}, \phi_j^* g_j F_-(N) g_j^{-1}\rangle \\ &\quad + \sum_{i=1}^{2k}(C_1\lambda_i^2 r^2 + C_2\lambda_i^2 d^3|\ln d| r^{-3} + \textit{higher order terms}),\end{aligned}$$

where F_+ *and* F_- *are the basic instanton and anti-instanton over* $\mathscr{R}^4$, *and* Q *is a positive constant. We now give some remarks about expanding the Yang-Mills functional in the power series in* λ.

Remark B.1. In this article the "interaction" among instantons, anti-instantons, and the background connection which is an isolated nonminimal solution to the Yang-Mills equations (i.e., the expansion of the Yang-Mills functional in the parameter λ) plays an important role. This kind of interaction phenomena of "mixed particles" has been used by C. H. Taubes for the Yang-Mills equations on S^4, where it has allowed him to prove that the Yang-Mills moduli space of SU(2) (or SU(3))-connections are path-connected spaces (cf. [20]). It has also been considered by Bahri and Coron (cf. [4]), where they used it to prove the existence theorem for the Yamabe equation on a certain domain in $\mathscr{R}^n$.

Appendix C. The proof of Proposition 5.4

$E_{2k\times 2k}$ is a $2k\times 2k$ matrix which is defined in §5. It is important for solving (4.35) by considering the eigenvalues of $E_{2k\times 2k}$. Our purpose here is to prove Proposition 5.4. To this end, the following result is used.

Lemma C.1. *Let $B_{n\times n}$ be an $n\times n$ matrix which obeys*

$$b_{ij}=\begin{cases} b_{j-i+1} & \text{if } i\le j\,,\\ b_{n+1+j-i} & \text{if } i>j\,,\end{cases} \tag{C.1}$$

where $\{b_i\}_{1\le i\le n}$ are n numbers. Then B_n has n eigenvalues with the expression

$$\lambda_i=\sum_{j=1}^{n} b_j e^{(i-1)(j-1)\theta(n)\sqrt{-1}}\,,\qquad 1\le i\le n\,, \tag{C.2}$$

where $\theta(n)=2\pi/n$.

Proof. Since the matrix $B_{n\times n}$ comes from S^1, let $\theta(n)=2\pi/n$. Think of the transformation

$$\begin{aligned} a_i &= \sum_{j=1}^{n} b_{ij}e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}\\ &= \sum_{j<i} b_{ij}e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}+\sum_{j\ge i} b_{ij}e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}\\ &= \sum_{j<i} b_{n+1+j-i}e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}+\sum_{j\ge i} b_{j-i+1}e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}\\ &= \left(\sum_{j=1}^{n} b_j e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}\right)e^{(\bar{i}-1)(i-1)\theta(n)\sqrt{-1}}\,. \end{aligned}$$

Hence,

$$V_i=(1\,,e^{(\bar{i}-1)\theta(n)\sqrt{-1}}\,,\cdots\,,e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{n-1}}\,,\cdots\,,e^{(\bar{i}-1)(n-1)\theta(n)\sqrt{-1}})$$

is an eigenvector of $B_{n\times n}$ with eigenvalue

$$\lambda_{\bar{i}}=\sum_{j=1}^{n} b_j e^{(\bar{i}-1)(j-1)\theta(n)\sqrt{-1}}\,,$$

which establishes (C.2) of Lemma C.1.

Proof of Proposition 5.4. $E_{2k\times 2k}$ may be viewed as a special example of Lemma C.1. Set

$$b_{\text{odd } i}=0\,,$$

$$b_{2l}=b_{2k+2-2l}=\begin{cases} 1/(2l-1)^4 & \text{if } (2l-1)\le 2[\tfrac12 d^{-2/5}]-1\,,\\ 0 & \text{if } (2l-1)> 2[\tfrac12 d^{-2/5}]-1\,.\end{cases} \tag{C.3}$$

Thus, the eigenvalues of $E_{2k\times 2k}$ can be written as

$$\lambda_i=\sum_{l=1}^{k} b_{2l}e^{(i-1)(2l-1)\pi\sqrt{-1}/k}\,,\qquad 1\le i\le 2k\,. \tag{C.4}$$

If k is even, it is not hard to see that $\lambda_{k/2+1} = 0$, so that $\det E_{2k\times 2k} = 0$. If k is odd, by direct calculation we have

(C.5)
$$\begin{aligned}\lambda_i &= \sum_{l=1}^{[d^{-2/5}/2]-1} \frac{e^{(i-1)(2l-1)\pi\sqrt{-1}/k} + e^{p(i-1(2l-1)\pi\sqrt{-1}/k}}{(2l-1)^4} \\ &= 2\sum_{l=0}^{[d^{-2/5}/2]-1} \frac{\cos(i-1)(2l+1)\pi/k}{(2l+l)^4}.\end{aligned}$$

Thus,

$$\lambda_1 = 2\sum_{l=0}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4}$$

is the maximum eigenvalue of $E_{2k\times 2k}$. Using the induction method, we can prove that

(C.6)
$$\left|\frac{\cos(2l+1)\theta}{\cos\theta}\right| \le 2l+1.$$

From this we get a priori estimates for eigenvalues of $E_{2k\times 2k}$:

(C.7)
$$|\lambda_i| \ge 2\left|\cos(i-1)\frac{\pi}{k}\right| \left\{1 - \sum_{l=1}^{[d^{-2/5}/2]-1} \frac{1}{(2l+1)^4}\right\} \ge \frac{3}{2}\left|\cos(i-1)\frac{\pi}{k}\right|.$$

Hence the proof of Proposition 5.4 is complete.

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References

[1] M. F. Atiyah, *Geometry of Yang-Mills fields*, Atti Accad. Naz. Lincei Scuola Norm. Sup. Pisa (1979).

[2] M. F. Atiyah & R. Bott, *The Yang-Mills equations over Riemannian surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1982) 523–615.

[3] M. F. Atiyah, N. Hitchin & I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978) 425–461.

[4] A. Bahri & J. M. Coron, *ON a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, preprint.

[5] M. Berger, P. Gaudnchon & E. Mazet, *Le spectre d'une variete riemannienne*, Lecture Notes in Math., Vol. 194, Springer, Berlin, 1971.
[6] J. P. Bourguinon & H. B. Lawson, Jr., *Stability and isolation for Yang-Mills theory*, Comm. Math. Phys. **79** (1982) 189–230.
[7] J. Cheeger & D. Ebin, *Comparisons theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.
[8] S. K. Donaldson, *An application of gauge theory to 4-dimensional topology*, J. Differential Geometry **18** (1983) 279–315.
[9] ——, *Connection, cohomology, and the intersection forms of 4-manifolds*, J. Differential Geometry **24** (1986) 275–341.
[10] D. Freed & K. K. Uhlenbeck, *Instantons and four-dimensional manifolds*, Springer, Berlin, 1984.
[11] P. H. Griffiths & J. Harris, *Principle of algebraic geometry*, Wiley-Interscience, New York, 1978.
[12] A. Grothendieck, *Sur la classification de fibre holomorphe sur sphere de Riemann*, Amer. J. Math. **79** (1957) 121–138.
[13] K. Kodaira & J. Morrow, *Complex manifolds*, Holt, Rinehart and Winston, New York, 1971.
[14] H. B. Lawson, Jr., *The theory of gauge fields in four-dimension*, Lecture Notes for NSF-CBMS Conference, University of California at Santa Barbara, 1983.
[15] T. Parker, *Unstable Yang-Mills fields*, preprint.
[16] L. M. Sibner, R. J. Sibner & K. K. Uhlenbeck, *Solutions to Yang-Mills equations which are not self-dual*, preprint.
[17] C. H. Taubes, *Self-dual connections on non-self-dual 4-manifolds*, J. Differential Geometry **17** (1982) 139–170.
[18] ——, *Stability in Yang-Mills theories*, Comm. Math. Phys. **91** (1983) 235–263.
[19] ——, *Self-dual connections on 4-manifolds with indefinite intersection matrix*, J. Differential Geometry **19** (1984) 517–560.
[20] ——, *Path-connected Yang-Mills moduli spaces*, J. Differential Geometry **19** (1984) 337–392.
[21] ——, *A framework for Morse theory for the Yang-Mills functional*, Invent. Math. **94** (1988) 347–402.
[22] K. K. Uhlenbeck, *Connections with L^p-bounds on curvature*, Comm. Math. Phys. **83** (1982) 31–42.
[23] ——, *Removable singularities for Yang-Mills fields*, Comm. Math. Phys. **83** (1982) 11–30.
[24] H. Wang, *The construction of isolated reducible* SU(2)*-connections over* $S^2 \times S^2$, preprint.
[25] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pacific J. Math. **121** (1986) 193–243.

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